DOUBTS ON GÖDEL'S INCOMPLETENESS THEOREMS

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K. Gödel's first and second incompleteness theorems state respectively that '*For any consistent formal theory that proves basic arithmetical truths, an arithmetical statement that is true but not provable in the theory can be constructed. That is, any theory capable of expressing elementary arithmetic cannot be both consistent and complete*.' and '*For any formal theory T including basic arithmetical truths and also certain truths about formal provability, T includes a statement of its own consistency if and only if T is inconsistent*.' However, B. Russell never appeared to be convinced by Gödel's proof; moreover, in this paper a formal theory T is presented that proves its consistency and completeness and that represents all the recursive relations. Accordingly, Gödel's theorems appear to be at least questionable and worthy of debate because of their basic importance in logic and in information physics as well, in spite of their general acceptance and of being considered something like religious dogmas excluding any possibility of discussion.

Keywords: Gödel's incompleteness theorems, sequence replacing, recursive relations, formal number theory

1. INTRODUCTION

1.1. Statement of the problem

K. Gödel has proved that '*For any consistent formal theory that proves basic arithmetical truths, an arithmetical statement that is true but not provable in the theory can be constructed. That is, any theory capable of expressing elementary arithmetic cannot be both consistent and complete*.' And, moreover, that '*For any formal theory T including basic arithmetical truths and also certain truths about formal provability, T includes a statement of its own consistency if and only if T is inconsistent*.' (Gödel, 1931). It is known that Bertrand Russell did not answer K. Gödel's communication when the latter sent him the proof of elementary arithmetic incompleteness. Why? Russell was judged harshly on that occasion, and perhaps even misjudged, as nobody considered the possibility that he had found in a way very evident to him so a gross mistake in Gödel's proof that he had thought the article unworthy of notice and even of an answer.

Is there any subtle fault, or even a blunder, in Gödel's proof, and where might it be lurking?

1.2. Recalling some logic concerning the Gödelian question

Let us recall that B. Russell distinguished a 'proposition' (today logicians prefer to use 'sentence') from a 'propositional function' (today logicians prefer to use 'sentential function'): e.g. "Peter is good" is a proposition and "*x* is good" is a propositional function where *x* is a variable whose field is the set of the possible subjects of the predicate “is good”. The elements of this set can obviously considered ‘constants’ as regard of the variable *x*. A proposition has to be true or false, a propositional function can be neither true nor false, but it assumes truth values if and only if we transform it in a proposition by substitution of its variables with a constant: e.g. “*x* is good” is neither true nor false but if we replace *x* with the constant “Peter” then we obtain “Peter is good” that has to have a truth value.

Let us recall also that the fundamental kernel of Gödel’s incompleteness proof is Gödelization: a natural number (Gödel’s number) corresponds one-to-one to any arithmetic expression. This mapping permits to build “*x* is not Gödel’s number of a demonstrable proposition” as arithmetic propositional function. Let [*x* is not Gödel’s number of a demonstrable proposition] represent the natural number that corresponds to “*x* is Gödel’s number of a demonstrable proposition” by Gödelization. As [*x* is not Gödel’s number of a demonstrable proposition] is a natural number and “*x* is not Gödel’s number of a demonstrable proposition” is an arithmetic propositional function, Gödel thought that expressions as “[*x* is not Gödel’s number of a demonstrable proposition] is not Gödel’s number of a demonstrable proposition” are propositions and he proved that they are undecidable bbecause of their self-referential meaning.

But, “[*x* is not Gödel’s number of a demonstrable proposition] is not Gödel’s number of a demonstrable proposition” is only syntactically a proposition. Semantically it is a self-referential propositional function whose variable *x*, in a lower linguistic level, has been hidden by Gödelization in the constant [*x* is not Gödel’s number of a demonstrable proposition]. Observe in fact that Gödelization transforms any arithmetic expression in numeric constants and in variables too. Thus self-referential propositional functions and self-referential propositions become indistinguishable and the former ones become obviously undecidable propositions because a propositional function can not have a truth value in itself. Perhaps B. Russell understood this semantic mistake, very serious for him, and thus he thought that Gödel’s incompleteness proofs were not worthy of answer.

1.3. B. Russell’s missing answer, and some objections to the general uncritical acceptance of Gödel’s theorems

B. Russell’s missing answer to K. Gödel was a serious mistake. Academic world did not notice the previously mentioned semantic mistake and it accepted instead Gödel’s proofs. The situation has become even worse when philosophers, religious, university men in career, politicians and so on have used incompleteness theorems, improperly too. Consequences of such theorems were drawn with applications to various fields, since the time of their formulations and up to the present days, and they have become something like religious dogmas excluding any possibility of discussion. For instance, E. Biedermann presented a paper at the Symposium on Gödel’s Theorem at Paris in 1991 (Biederman, 1991) with **very heavy objections** against Gödel’s proofs. The paper was fully ignored at the Symposium: no serious objections were raised, but in spite of that the paper was not published in the proceedings. We report Biedermann’s preprint partially because it is very interesting: … ‘*Gödel introduced what is now known as the Gödel numbering technique that leads to the arithmetization of mathematics, where every formula* R *is represented by a specific natural number, it’s Gödel number. By means of some 45 recursive definitions, including the here crucially important substitution function* sub(m,19,Z(n*)), Gödel then constructs his undecidable formula*’ … ‘ G : (x)(~ Dem(x, sub(n, 19, Z(n)))’ … ‘*which results from the formula with the one variable* y’…‘ F : (x) ~ Dem(x, sub(y, 19, Z(y)))’ …‘*through substitution of the variable* y *by the number* n*, which is here assumed to be the Gödel number of formula* F*, what leads (when the* 19 *is assumed to be the Gödel number for the variable* y*) to the claim that the* sub(n,19,Z(n)) *in* G *be a definition of the Gödel number of just this formula. So, the interpretation of* G *then reads:* “no number *x* satisfies the proof relation to Gödel’s number of formula *G.”.*’ … ‘*Now, formula* F *certainly is a correct arithmetical formula of the one variable* y *(it could easily be brought into a shape with only one occurrence of the variable) that appears somewhere in our list of all the formulas of all the formulas of the variable* y*. In fact, in that list, there is an infinite multitude of similar formulas in which the number* 19 *is replaced by any other natural number. As such, as simple arithmetical formulas, any one of them is as good (or as useless) as all the others. No question about that!*’ … ‘*However, the above argument, as well as Gödel’s detailed proof to his theorem, hinges on interpretation of the number* 19 *as Gödel’s number of the variable* y*. This yields for the* sub(y,19,Z(y)) *the interpretation:*

*"*the Gödel number of that formula that results form the formula with Gödel number *y* through substitution of the variable *y* by the number *y*.*" Evidently, this interpretation of the* y *as a designation of the Gödel number of a formula* F(y) *presupposes as given from the outset, what then is claimed to be brought about by the introduction of the substitution function, i. e. the identity of the Gödel number of a formula with the argument in that same formula. This assignment of two divergent meanings to one and the same symbol* y *certainly constitutes a gross violation of the PM rule to preserve the identity of a variable throughout a given context. The claim that formulas* F *and* G *contain the* y *in only one meaning is easily countered: the simultaneous occurrence of the symbol* y *side by side with its encoding* 19 *is an unacceptable constellation;* **nowadays even the youngest programmer realizes that he should never mix two levels of code within one statement; his program would never work.'**

The emergence of *incursive*, *hyperincursive* and *anticipatory mathematics* in the early 90's (Dubois, 2000 for an in-depth review) and its influence on Logic (Dubois, 1996) and Cybernetics (Davidson, Astor and Ekdahl, 1994) put in crisis Gödel's incompleteness proof on another side. Any anticipatory algorithm (Dubois, 1998) is completely reversible because any step contains implicitly information of previous and successive steps. Thus if we build Gödelization in form of anticipatory algorithm (it has been proven that any algorithm can be replaced by an opportune anticipatory algorithm (Grappone, 1999)) then the numeric constant [*x* is not Gödel's number of a demonstrable proposition] conserves information that *x* is a variable and so "[*x* is not Gödel's number of a demonstrable proposition] is not Gödel's number of a demonstrable proposition" keeps information as a propositional function and hence it cannot be an undecidable proposition (Grappone, 1999). As a confirmation of this, consider that Dubois (Dubois, 1998) observes: … '*The Boolean tables of the Hyperincursive neuron is totally different from the Recursive neuron. The recursive neuron is a classical feedback system, with a time lag of the output to feed the input: there are two solutions, a fixed point x=0, y=1, and a bifurcation x=1, y=0,1,0,1,0,1… The Hyperincursive neuron is in fact an incursive neuron because there is only one solution, a fixed point: x=0, y=1. The hypeincursive equation avoids the Gödelian undecidability: indeed, for x=1, we have y=0 if y=1 and y=1 if y=0.*'

This paper tries a falsification of Gödel's incompleteness theorems finally by direct construction of a formal theory *T* that proves its completeness and its consistency and that can represent all the elementary arithmetic as such a formal theory has not to exist for these theorems. Reference can be made to the vast literature of Logic for the basic notions, procedures and symbols employed throughout this paper (Herbrand, 1930; Lewis and Langford, 1959; Malatesta, 1997, Mendelson, 1964, Nicod, 1917).

2. FORMAL THEORY *T*

2.1. Definition of Formal Theory *T*

2.1.1. *T* Alphabet

2.1.1.1. p, $_1$, $_2$, $_3$, …, D and round brackets are exclusively the signs of *T* well formed formulas (abbreviate with *wffs*, singular *wff*).

2.1.1.2. $\mapsto$ and comma are exclusively the deduction meta-linguistic signs of *T* inferences.

2.1.1.3. **R**, **I**, **E** are exclusively meta-predicates of *T* well formed formulas.

2.1.2. *T* Meta-syntax

2.1.2.1. If α, β, γ, … are *T wffs* then $\beta, \gamma, \ldots \mapsto \alpha$ represents an inference scheme in *T*.

2.1.2.2. If α, β, γ are *T wffs* then $\mathbf{R}\alpha\beta\gamma$ is the *T wff* that is obtained from γ by replacing every its occurrence of β with α.

2.1.2.3. If α is *T wff* then $\mathbf{I}\alpha$ is a *T wff* that is not sub-formula of another *T wff*.

2.1.2.4. If α and β are *T wffs* then $\mathbf{E}\alpha\beta$ is a *T wff* β where α does not occur.

2.1.2.5. There are not other formalized meta-linguistic expressions for *T*.

2.1.3. *T* Syntax

2.1.3.1. $p_1, p_2, p_3, \ldots$ are *T wffs*.
2.1.3.2. (D) is a *T wff*.
2.1.3.3. If $\alpha, \beta, \gamma, \ldots$ are *T wffs* then $(D\alpha\beta\gamma\ldots)$ is *T wff*.
2.1.3.4. 2.1.3.1., …, 2.1.3.3. are only the rules to build *T wffs*.

2.1.4. *T* Axiom

2.1.4.1. $(D(D))$.
2.1.4.2. There are not other *T* axioms.

2.1.5. *T* Inference Rules

2.1.5.1. If α, β, γ are *T wffs* then $\mathbf{R}(D\ldots\beta\ldots\gamma\ldots)(D\ldots\gamma\ldots\beta\ldots)\alpha \mapsto \alpha$ (*D* commutative property).
2.1.5.2. If α is *T wff* then $\mathbf{R}(D(D))(D\ldots(D)\ldots)\alpha \mapsto \alpha$ (bracketed *D* expansion).
2.1.5.3. If α and β are *T wffs* then $\mathbf{R}(D\ldots\beta\ldots\gamma\ldots)(D\ldots(D(D\beta\ldots\gamma))\ldots)\alpha \mapsto \alpha$ (internal double *D* introduction). A particular case is $\mathbf{R}(D\ldots\ldots)(D\ldots(D(D))\ldots)\alpha \mapsto \alpha$.
2.1.5.4. If $\alpha, \beta, \gamma, \delta$ are *T wffs* then $\mathbf{R}(D\ldots\beta\ldots\gamma\delta\ldots)(D\ldots\beta\ldots\gamma\beta\delta\ldots)\alpha \mapsto \alpha$ (*D* argument duplication).
2.1.5.5. If $\alpha, \beta, \gamma, \delta$ are *T wffs* then $\mathbf{R}(D\ldots\beta\ldots(D\ldots\gamma\delta\ldots)\ldots)(D\ldots\beta\ldots(D\ldots\gamma\beta\delta\ldots)\ldots)\alpha \mapsto \alpha$ (nested *D* argument duplication).
2.1.5.6. If $\alpha, \beta, \gamma, \delta, \varepsilon$ are *T wffs* then $\mathbf{R}(D(D(D\ldots(D\beta\ldots\gamma)\ldots)(D\ldots(D\delta\ldots\varepsilon)\ldots)))(D\ldots(D\beta\ldots\gamma\delta\ldots\varepsilon)\ldots)\alpha \mapsto \alpha$ (*D* contraction).
2.1.5.7. If α, β, γ are *T wffs* then $\mathbf{R}(D\ldots(D\beta\ldots\gamma)\ldots)(D\ldots(D\beta\ldots\gamma)(D\beta\ldots\gamma\delta\ldots\varepsilon)\ldots)\alpha \mapsto \alpha$ (more weak *D* introduction).
2.1.5.8. *T* can use inferences with a premise only and a conclusion only.

2.1.6. *T* Meta-inference Schemes

2.1.6.1. If $\alpha, \beta, \gamma, \ldots$ are *T wffs* then $(D\beta\gamma\ldots(D\alpha)) \mapsto (\beta, \gamma, \ldots \mapsto \alpha)$ (deduction meta-inference).
2.1.6.2. If $\alpha, \beta, \gamma, \delta, \varepsilon, \ldots$ are *T wffs* then $(\alpha \mapsto \gamma, \gamma \mapsto \varepsilon, \ldots, \beta \mapsto \delta) \mapsto (\alpha \mapsto \varepsilon, \ldots, \beta \mapsto \delta)$.
2.1.6.3. *T* can not use other meta-inferences.

2.2. Proof that Standard Sentence Logic is a *T* Model

2.2.1. Proof of the *T* Inclusion in Some Formal Theories Whose Model is Standard Sentence Logic

2.2.1.1. Assume standard sentence logic (abbreviate it with *SSL*), standard tautology calculation and a standard language: $\neg p_1$ is the negation of p_1, $p_1 \Rightarrow p_2$ is "if p_1 then p_2", $p_1 \vee p_2$ is "p_1 and/or p_2", $p_1 \uparrow p_2$ is $\neg p_1 \vee \neg p_2$, $p_1 \wedge p_2$ is "p_1 and p_2", "$p_1 \equiv p_2$ is "p_1 if and only if p_2".

2.2.1.2. For all the expressions α, put $\mathbf{R}(p_1 \wedge \neg p_1)(D)\alpha \mapsto \alpha$.
2.2.1.3. For all the expressions α, β, put $\mathbf{R}(\neg\beta)(D\beta)\alpha \mapsto \alpha$.
2.2.1.4. For all the expressions $\alpha, \beta, \gamma, \delta, \ldots$, put $\mathbf{R}(\neg\beta \vee \neg\gamma \vee \neg\delta \vee \ldots)(D\beta\gamma\delta\ldots)\alpha \mapsto \alpha$.

2.2.1.5. The *T* axiom $(D(D))$ is deduced by 2.2.1.2. from $(D(p_1 \wedge \neg p_1))$ that is deduced by 2.2.1.3. from $\neg(p_1 \wedge \neg p_1)$ that is a tautology.

2.2.1.6. The *T* inference rule $\mathbf{R}(D\ldots\beta\ldots\gamma\ldots)(D\ldots\gamma\ldots\beta\ldots)\alpha \mapsto \alpha$ is deduced by 2.2.1.4. from $\mathbf{R}(\ldots\vee\neg\beta\vee\ldots\vee\neg\gamma\vee\ldots)(\ldots\vee\neg\gamma\vee\ldots\vee\neg\beta\vee\ldots)\alpha \mapsto \alpha$ that is a *SSL* valid inference rule.

2.2.1.7. The *T* inference rule $\mathbf{R}(D(D))(D\ldots(D)\ldots)\alpha \mapsto \alpha$ is deduced by 2.2.1.2. from $\mathbf{R}(D(p_1\wedge\neg p_1))(D\ldots(p_1\wedge\neg p_1)\ldots)\alpha \mapsto \alpha$ that is deduced by 2.2.1.3. from $\mathbf{R}\neg(p_1\wedge\neg p_1)(D\ldots(p_1\wedge\neg p_1)\ldots)\alpha \mapsto \alpha$ that is deduced by 2.2.1.4. from $\mathbf{R}\neg(p_1\wedge\neg p_1)(\ldots\vee\neg(p_1\wedge\neg p_1)\vee\ldots)\alpha \mapsto \alpha$ that is a *SSL* valid inference rule.

2.2.1.8. The *T* inference rule $\mathbf{R}(D\ldots\beta\ldots\gamma\ldots)(D\ldots(D(D\beta\ldots\gamma))\ldots)\alpha \mapsto \alpha$ is deduced by 2.2.1.3. from $\mathbf{R}(D\ldots\beta\ldots\gamma\ldots)(D\ldots\neg(D\beta\ldots\gamma)\ldots)\alpha \mapsto \alpha$ that is deduced by 2.2.1.4. from $\mathbf{R}(\ldots\vee\neg\beta\vee\ldots\vee\neg\gamma\vee\ldots)(\ldots\vee\neg\neg(\neg\beta\vee\ldots\vee\neg\gamma)\vee\ldots)\alpha \mapsto \alpha$ that is a *SSL* valid inference rule.

2.2.1.9. The *T* inference rule $\mathbf{R}(D\ldots\beta\ldots\gamma\delta\ldots)(D\ldots\beta\ldots\gamma\beta\delta\ldots)\alpha \mapsto \alpha$ is deduced by 2.2.1.4. from $\mathbf{R}(\ldots\vee\neg\beta\vee\ldots\vee\neg\gamma\vee\neg\delta\vee\ldots)(\ldots\vee\neg\beta\vee\ldots\vee\neg\gamma\vee\neg\beta\vee\neg\delta\vee\ldots)\alpha \mapsto \alpha$ that is a *SSL* valid inference rule.

2.2.1.10. The *T* inference rule $\mathbf{R}(D\ldots\beta\ldots(D\ldots\gamma\delta\ldots)\ldots)(D\ldots\beta\ldots(D\ldots\gamma\beta\delta\ldots)\ldots)\alpha \mapsto \alpha$ is deduced by 2.2.1.4. from $\mathbf{R}(\ldots\vee\neg\beta\vee\ldots(\ldots\vee\neg\gamma\vee\neg\delta\vee\ldots)\ldots)(\ldots\vee\neg\beta\vee\ldots(\ldots\vee\neg\gamma\vee\neg\beta\vee\neg\delta\vee\ldots)\ldots)\alpha \mapsto \alpha$ that is a *SSL* valid inference rule.

2.2.1.11. The *T* inference rule $\mathbf{R}(D(D(D\ldots(D\beta\ldots\gamma)\ldots)(D\ldots(D\delta\ldots\varepsilon)\ldots)))(D\ldots(D\beta\ldots\gamma\delta\ldots\varepsilon)\ldots)\alpha \mapsto \alpha$ is deduced by 2.2.1.3. from $\mathbf{R}\neg(D(D\ldots(D\beta\ldots\gamma)\ldots)(D\ldots(D\delta\ldots\varepsilon)\ldots))(D\ldots(D\beta\ldots\gamma\delta\ldots\varepsilon)\ldots)\alpha \mapsto \alpha$ that is deduced by 2.2.1.4. from $\mathbf{R}\neg(\neg(\ldots\vee\neg(\neg\beta\vee\ldots\vee\neg\gamma)\vee\ldots)\vee\neg(\ldots\vee\neg(\neg\delta\vee\ldots\vee\neg\varepsilon)\vee\ldots))(\ldots\vee\neg(\neg\beta\vee\ldots\vee\neg\gamma\vee\neg\delta\vee\ldots\vee\neg\varepsilon)\vee\ldots)\alpha \mapsto \alpha$ that is a *SSL* valid inference rule.

2.2.1.12. The *T* inference rule $\mathbf{R}(D\ldots(D\beta\ldots\gamma)\ldots)(D\ldots(D\beta\ldots\gamma)(D\beta\ldots\gamma\delta\ldots\varepsilon)\ldots)\alpha \mapsto \alpha$ is deduced by 2.2.1.4. from $\mathbf{R}(\ldots\vee\neg(\neg\beta\vee\ldots\vee\neg\gamma)\vee\ldots)(\ldots\vee\neg(\neg\beta\vee\ldots\vee\neg\gamma)\vee\neg(\neg\beta\vee\ldots\vee\neg\gamma\vee\neg\delta\vee\ldots\vee\neg\varepsilon)\vee\ldots)\alpha \mapsto \alpha$ that is a *SSL* valid inference rule.

2.2.1.13. The *T* meta-inference scheme $(D\beta\gamma\ldots(D\alpha)) \mapsto (\beta, \gamma, \ldots \mapsto \alpha)$ is deduced by 2.2.1.3. from $(D\beta\gamma\ldots\neg\alpha) \mapsto (\beta, \gamma, \ldots \mapsto \alpha)$ that is deduced by 2.2.1.4. from $(\neg\beta\vee\neg\gamma\vee\ldots\vee\neg\neg\alpha) \mapsto (\beta, \gamma, \ldots \mapsto \alpha)$, i.e. $(\neg\beta\vee\neg\gamma\vee\ldots\vee\alpha) \mapsto (\beta, \gamma, \ldots \mapsto \alpha)$, i.e. $(\beta\Rightarrow(\gamma\Rightarrow(\ldots(\ldots\Rightarrow\alpha)\ldots))) \mapsto (\beta, \gamma, \ldots \mapsto \alpha)$ (cf the sequence implication, see Malatesta, 1997) that is a *SSL* valid inference scheme that is obtained by repeated applications of Herbrand's deduction meta-theorem for *SSL*.

2.2.1.14. The *T* meta-inference scheme $(\alpha \mapsto \gamma, \gamma \mapsto \varepsilon, \ldots, \beta \mapsto \delta) \mapsto (\alpha \mapsto \varepsilon, \ldots, \beta \mapsto \delta)$ is a *SSL* valid inference scheme too.

2.2.1.15. 2.2.1.5., …, 2.2.1.14 permits us to affirm that the *T* axiom and all the *T* theorems are *SSL* axioms or theorems, i.e. tautologies and that all the *T* inferences and meta-inferences are valid in *SSL* too. Thus we can conclude that the formal theory *T* is included in some formal theories that has *SSL* as model.

2.2.2. Proof of Inclusion of Some Formal Theories Whose Model is Standard Sentence Logic in *T*

2.2.2.1. Consider Nicod's formal theory *NT* for *SSL* (Nicod, 1917) We choose it because it has only an axiom scheme and only an inference scheme. Obviously some formal theories that has *SSL* as model are included in *NT*. To realize our purpose we can so prove that *NT* is included in *T*. To obtain this achievement we build the *T* proofs of *NT* axiom scheme and of *NT* inference scheme.

2.2.2.2. For all the expressions α, β, γ put $\mathbf{R}(D\beta\gamma)(\beta\uparrow\gamma)\alpha \mapsto \alpha$.

2.2.2.3. For all the *SSL wffs* $\alpha, \beta, \gamma, \delta, \varepsilon, \ldots$, *NT* axiom scheme is $((\alpha\uparrow(\beta\uparrow\gamma))\uparrow((\delta\uparrow(\delta\uparrow\delta))\uparrow((\varepsilon\uparrow\beta)\uparrow((\alpha\uparrow\varepsilon)\uparrow(\alpha\uparrow\varepsilon)))))$. This one is deduced by 2.2.2.2. from $(D(D\alpha(D\beta\gamma))(D(D\delta(D\delta\delta))(D(D\varepsilon\beta)(D(D\alpha\varepsilon)(D\alpha\varepsilon)))))$ that is deduced by 2.1.5.4. from

$(D(D\alpha(D\beta\gamma))(D(D\delta(D\delta))(D(D\varepsilon\beta)(D(D\alpha\varepsilon)))))$ that is deduced by 2.1.5.5. from
$(D(D\alpha(D\beta\gamma))(D(D\delta(D))(D(D\varepsilon\beta)(D(D\alpha\varepsilon)))))$ that is deduced by 2.1.5.2. from
$(D(D\alpha(D\beta\gamma))(D(D(D))(D(D\varepsilon\beta)(D(D\alpha\varepsilon)))))$ that is deduced by 2.1.5.3. from
$(D(D\alpha(D\beta\gamma))(D)(D(D\varepsilon\beta)\alpha\varepsilon)))$ that is deduced by 2.1.5.2. from $(D(D))$ that is the *T* axiom.

2.2.2.4. For all the *SSL wffs* $\alpha, \beta, \gamma, \delta, \varepsilon, \ldots$, *NT* inference scheme is $(\alpha\uparrow(\beta\uparrow\gamma)), \alpha \mapsto \gamma$. This one is deduced by 2.2.2.2. from $(D\alpha(D\beta\gamma)), \alpha \mapsto \gamma$ that is deduced by 2.1.6.1. from $(D(D\alpha(D\beta\gamma))\alpha(D\gamma))$ that is deduced by 2.1.5.1. from $(D\alpha(D\alpha(D\beta\gamma))(D\gamma))$ that is deduced by 2.1.5.5. from $(D\alpha(D(D\beta\gamma))(D\gamma))$ that is deduced by 2.1.5.3. from $(D\alpha\beta\gamma(D\gamma))$ that is deduced by 2.1.5.5. from $(D\alpha\beta\gamma(D))$ that is deduced by 2.1.5.2. from $(D(D))$ that is the *T* axiom.

2.2.2.5. The proven reciprocal inclusion between *T* and some formal theories whose model is *SSL* proves us that *SSL* is a model for *T*. This achievement permits us to use *SSL* connectives in *T* if, e.g., there is the necessity of clearness.

2.3. Consistency and Completeness of *T*: Self-proofs

2.3.1. Construction of of Meta-predicate "It can proven in *T* that" in *T*

2.3.1.1. Consider that *T* has an axiom only: $(D(D))$.
2.3.1.2. Consider that all the inferences and the meta-inference that *T* can use have got a premise only and a conclusion only.
2.3.1.3. 2.3.1.1 and 2.3.1.2 permit us to affirm that, if $\alpha, \beta, \gamma, \delta, \varepsilon, \ldots$ are *T wffs* then every *T* proof of a *T wff* δ has got the form $(D(D)) \mapsto \alpha, \alpha \mapsto \gamma, \gamma \mapsto \varepsilon, \ldots, \beta \mapsto \delta$ always.
2.3.1.4. $(D(D)) \mapsto \delta$ is always deduced from $(D(D)) \mapsto \alpha, \alpha \mapsto \gamma, \gamma \mapsto \varepsilon, \ldots, \beta \mapsto \delta$ by repeated applications of 2.1.6.2.
2.3.1.5. Consider that if $(D(D)) \mapsto \delta$ is directly obtained without 2.1.6.2. then it is a *T* proof however because $(D(D))$ is the *T* axiom.
2.3.1.6. We can deduce from 2.3.1.4 and 2.3.1.5 that every *T wff* α can be proven in *T* if and only if $(D(D)) \mapsto \alpha$.
2.3.1.7. As $(D(D)) \mapsto \alpha$ is deduced by 2.1.6.1. from $(D(D(D))(D\alpha))$ that is deduced by 2.1.5.3. from $(D(D\alpha))$, we can assume $(D(D\alpha))$ as "It can be proven in *T* that α".
2.3.1.8. Thus let **Dim**(α), i.e. "It can be proven in *T* that α", be $(D(D\alpha))$.

2.3.2. Construction and Proof of Sentence "*T* is consistent" in *T*

2.3.2.1. Observe that the sentence "*T* is consistent" is equivalent to "For every *T wff* α, α can not be proven in *T* and/or the negation of α can not be proven in *T*". We can represent this last sentence in *T* by **Dim**$(\alpha)\uparrow$**Dim**$(\alpha\uparrow\alpha)$ (see 2.2.2.5.)
2.3.2.2. **Dim**$(\alpha)\uparrow$**Dim**$(\alpha\uparrow\alpha)$ is deduced by 2.3.1.8. from $(D(D\alpha))\uparrow(D(D(\alpha\uparrow\alpha)))$ that is deduced by 2.2.2.2. from $(D(D(D\alpha))(D(D(D\alpha\alpha))))$ that is deduced by 2.1.5.4. from $(D(D(D\alpha))(D(D(D\alpha))))$ that is deduced by 2.1.5.5. from $(D(D(D\alpha))(D))$ that is deduced by 2.1.5.2. from $(D(D))$ that is the *T* axiom.

2.3.3. Construction and Proof of Sentence "*T* is complete" in *T*

2.3.3.1. Observe that the sentence "*T* is complete" is equivalent to "For every *T wff* α, α deduces in *T* that α can be proven in *T*". We can represent this one in *T* by $\alpha \mapsto$ **Dim**(α) that is deduced by 2.1.6.1. from ($D\alpha$(D**Dim**(α))). This last one is the *T wff* that represents "*T* is complete" in *T*.

2.3.3.2. ($D\alpha$(D**Dim**(α))) is deduced by 2.3.1.8. from ($D\alpha(D(D(D\alpha)))$) that is deduced by 2.1.5.3. from ($D\alpha(D\alpha)$) that is deduced by 2.1.5.5. from ($D\alpha(D)$) that is deduced by 2.1.5.2. from ($D\alpha(D)$) that is the *T* axiom.

2.4. Representation of All the Recursive Relations (i.e. of Basic Arithmetic) in *T*

2.4.1. Definition of *wff* Generic Relation in *T*

2.4.1.1. Let ($\alpha\beta\gamma\delta\ldots$) be a *T wff* α where eventual occurrences of the *T wff* β, γ, δ, … in α as sub-formulas have no overlapping among them and the eventual most left occurrence of β is more left than the eventual most occurrence of γ, the eventual most left occurrence of γ is more left than the eventual most left occurrence of δ and so on. Observe that so ($\alpha\ldots\beta\ldots\gamma\ldots$) is not equivalent in general to ($\alpha\ldots\gamma\ldots\beta\ldots$).

2.4.1.2. Let ($\alpha\beta\ldots\gamma\delta\varepsilon\ldots\zeta$) be **S**$\delta\eta$($\alpha\beta\ldots\gamma\eta\varepsilon\ldots\zeta$).

2.4.1.3. Let α be a *T wff* and $\beta\ldots\gamma$ be a succession of *n T wffs*. As *SSL* is a *T* model, ($\alpha\beta\ldots\gamma$) can be interpreted as a sentence whose truth-value depends from $\beta\ldots\gamma$. As ($\alpha\beta\ldots\gamma$) can assume two truth-values only in *SSL*, it makes a binary partition on the set of all the successions of *n T wffs* that can replace $\beta\ldots\gamma$ in it. Thus it define a relation among sentences and so *wff* relations are defined in *T* too.

2.4.2. Definition of *wff* Quantifiers in *T*

2.4.2.1. Let α, β, γ, δ, ε, ζ, … be *T wffs*. Consider $(\alpha\beta\ldots\gamma\delta\varepsilon\ldots\zeta)\wedge(\alpha\beta\ldots\gamma\neg\delta\varepsilon\ldots\zeta)$. Observe that the truth function of δ is completely irrelevant in the calculation of the truth function of $(\alpha\beta\ldots\gamma\delta\varepsilon\ldots\zeta)\wedge(\alpha\beta\ldots\gamma\neg\delta\varepsilon\ldots\zeta)$ by standard tautology calculus. E.g., consider the *wff* $\alpha\vee\beta$. It can be written by 2.4.1.1. as $(\alpha\vee\beta\alpha\beta)$. Consider $(\alpha\vee\beta\alpha\beta)\wedge(\alpha\vee\neg\beta\alpha\neg\beta)$ now. It can written by 2.4.1.1. as $((\alpha\vee\beta)\wedge(\alpha\vee\neg\beta)\alpha\beta)$, i.e. as $((\alpha\wedge\alpha)\vee(\alpha\wedge\neg\beta)\vee(\beta\wedge\alpha)\vee(\beta\wedge\neg\beta)\alpha\beta)$, i.e. as $(\alpha\vee(\alpha\wedge\neg\beta)\vee(\alpha\wedge\beta)\alpha\beta)$, i.e. as $(\alpha\vee\alpha\wedge(\neg\beta\vee\beta)\alpha\beta)$, i.e. as $(\alpha\vee\alpha\alpha\beta)$, i.e. as $(\alpha\alpha\beta)$ where β is trivially irrelevant in the calculation of the truth function of α.

2.4.2.2. As α is irrelevant in $(\beta\ldots\gamma\alpha\delta\ldots)\wedge(\beta\ldots\gamma\neg\alpha\delta\ldots)$, for **every** *T wff* ε that respects the conditions of 2.4.1.1., $(\beta\ldots\gamma\alpha\delta\ldots)\wedge(\beta\ldots\gamma\neg\alpha\delta\ldots)\Rightarrow(\beta\ldots\gamma\varepsilon\delta\ldots)$. This achievement permits us to rewrite $(\beta\ldots\gamma\alpha\delta\ldots)\wedge(\beta\ldots\gamma\neg\alpha\delta\ldots)$ as $\prod\alpha(\beta\ldots\gamma\alpha\delta\ldots)$, i.e. as if we consider $(\beta\ldots\gamma\alpha\delta\ldots)\wedge(\beta\ldots\gamma\neg\alpha\delta\ldots)$ as the finite binary resolution of the universal quantifier $\prod\alpha$ that is applied to the sub-formula α on $(\beta\ldots\gamma\alpha\delta\ldots)$. Lewis and Langford have effectively proven the possibility to build a valid quantification theory by a two-constant resolution of any quantified variable (see Lewis and Langford, 1959).

2.4.2.3. A trivial consequence of 2.4.2.2. is to define the particular quantification $\sum\alpha(\beta\ldots\gamma\alpha\delta\ldots)$ as $\neg\prod\alpha\neg(\beta\ldots\gamma\alpha\delta\ldots)$, i.e. $(D\prod\alpha(D(\beta\ldots\gamma\alpha\delta\ldots)))$.

2.4.3. Definition of *wff* Projection Relation in *T*

2.4.3.1. Let α, β, γ, δ, ε, ζ, … be *T wffs*. Let $=\alpha\beta$ be tautology if and only if $\alpha\equiv\beta$ is tautology, otherwise let it be a contradiction. Thus we can consider $=\alpha\beta$ the *wff* identity relation between α and β. Given a sentence **IE**$\alpha(\beta\ldots=\gamma\delta\ldots)$ (see 2.1.2.) we have three case only:

2.4.3.1.1. $=\gamma\delta$ is tautology and so γ replaces δ in **IE**$\alpha(\beta\ldots=\gamma\delta\ldots)$: this case is represented in *T* by **S**$\gamma\delta$**S**$(D(D))(\gamma=\delta)$**IE**$\alpha(\beta\ldots=\gamma\delta\ldots)$.

2.4.3.1.2. =γδ is tautology and so δ replaces γ in **IE**α(β…=γδ…): this case is represented in *T* by **S**δγ**S**(*D*(*D*))(γ=δ)**IE**α(β…=γδ…).

2.4.3.1.3. =γδ is contradiction and so γ∧¬δ replaces γ and δ∧¬γ replaces δ in **IE**α(β…=γδ…): this case is represented in *T* by **S**(*D*(*D*γ(*D*δ)))α**S**(*D*(*D*δ(*D*γ)))δ**S**αγ**S**(*D*)(γ=δ)**IE**α(β…=γδ…). Observe that α has got a double-swap buffer function.

Thus let **IE**α(β…=γδ…) be **S**γδ**S**(*D*(*D*))(γ=δ)**IE**α(β…=γδ…)∨**S**δγ**S**(*D*(*D*))(γ=δ)**IE**α(β…=γδ …)∨**S**(*D*(*D*γ(*D*δ)))α**S**(*D*(*D*δ(*D*γ)))δ**S**αγ**S**(*D*)(γ=δ)**IE**α(β…=γδ…). As the second member of this defintion does not contain =γδ, it remains defined too.

2.4.3.2. Let α, β, γ, δ, ε, ζ, η, … be *T wffs*. Let (**u**βδζ…**;**γεη…α) be (=γαβδζ…γεη…α), i.e. =γα. So we can consider (**u**βδζ…;γεη…α) the projection relation of recursion theory (abbreviate with *RT*) that is represented among the *T wffs*.

2.4.4. Definition of *wff* Successor Relation in *T*

2.4.4.1. If α, β, γ, δ, ε, ζ, … are *T wffs*, then let (**s**βγδεζ…) be defined by the following inference rules (see Peano’s axioms):

2.4.4.1.1. **S**(**s**βγδεζ…)((**s**βγ)(**s**γδ)(**s**δε)(**s**εζ)…)α ↦ α;

2.4.4.1.2. **S**((**s**βγ)(**s**γδ)(**s**δε)(**s**εζ)…)(**s**βγδεζ…)α ↦ α;

2.4.4.1.3. **S**((**s**…β…β…)(*D*))(**s**…β…β…)α ↦ α;

2.4.4.1.4. **S**((**s**…β(*D*)…)(*D*))(**s**…β(*D*)…)α ↦ α;

2.4.4.1.5. **S**((**s**…αβ…)(**s**…αγ…)=βγ)((**s**…αβ…)(**s**…αγ…))α ↦ α;

2.4.4.1.6. **S**((**s**…αβ…)(**s**…γβ…)=αγ)((**s**…αβ…)(**s**…γβ…))α ↦ α.

2.4.4.2. So we can consider (**s**αβ) the successor relation of *RT* that is represented among the *T wffs* and (**s**αβγδε…) a chain succession.

2.4.5. Definition of *wff* Zero Relation in *T*

2.4.5.1. If α and β are *T wffs*, then let **z**α be ∏β(*D*(**s**βα)). So we can consider **z**α the zero relation of *RT* that is represented among the *T wffs*.

2.4.6. Definition of *wff* Arithmetic sum Relation in *T*

2.4.6.1. If α, β, γ, δ, ε, ζ, … are *T wffs*, then let +βγδ be defined by the following inference rules (see Peano’s axioms):

2.4.6.1.1. **S**+βγδ+γβδα ↦ α;

2.4.6.1.2. **S**(+βγδ**z**γ=αβδ)(+βγδ**z**γ)α ↦ α;

2.4.6.1.3. **S**(+βγδ=αβδ**z**γ)(+βγδ=αβδ)α ↦ α;

2.4.6.1.4. **S**(+βγδ+βεζ(**s**γε)(**s**δζ))(+βγδ+βεζ(**s**γε))α ↦ α;

2.4.6.1.5. **S**(+βγδ+βεζ(**s**δζ)(**s**γε))(+βγδ+βεζ(**s**δζ))α ↦ α.

2.4.6.2. So we can consider +βγδ the arithmetic sum relation of *RT* that is represented among the *T wffs*.

2.4.7. Definition of *wff* Arithmetic Product Relation in *T*

2.4.7.1. If α, β, γ, δ, ε, ζ, … are *T wffs*, then let ·βγδ be defined by the following inference rules (see Peano’s axioms):

2.4.7.1.1. **S**·βγδ·γβδα ↦ α;

2.4.7.1.2. **S**(·βγδ**z**γ**z**δ)(·βγδ**z**γ)α ↦ α;

2.4.7.1.3. **S**(·βγδ**z**δ(*D*(*D***z**β)(*D***z**γ)))(·βγδ**z**δ)α ↦ α;

2.4.7.1.4. let ·αβγ·αδε(**s**βδ) be ·αβγ·αδε(**s**βδ)+αγε;

2.4.7.1.5. **S**(·βγδ·βεζ(**s**γε)+βδζ)(·βγδ·βεζ(**s**γε))α ↦ α;

2.4.7.1.6. **S**(·βγδ·βεζ+βδζ(**s**γε))(·βγδ·βεζ+βδζ)α ↦ α.

2.4.7.2. So we can consider $\boldsymbol{\cdot}\beta\gamma\delta$ the arithmetic product relation of *RT* that is represented among the *T wffs*.

2.4.8. Definition of *wff* "lesser than" Relation in *T*

2.4.8.1. If α, β, γ are *T wffs*, then let $\boldsymbol{<}\alpha\beta$ be $\sum\gamma(D(D(\boldsymbol{+}\alpha\gamma\beta)(D\mathbf{z}\gamma)))$. So we can consider $\boldsymbol{<}\alpha\beta$ the "lesser than" relation of *RT* that is represented among the *T wffs*.

2.4.9. Definition of *wff* Gödel's β Function Relation in *T*

2.4.9.1. If α, β, γ, δ, ε, ζ, η, ϑ, ι, … are *T wffs*, then let $\boldsymbol{\beta}\alpha\beta\gamma\delta$ be $\sum\varepsilon\sum\zeta\sum\eta\sum\vartheta\sum\iota(D(D{+}\zeta\delta\alpha{\cdot}\eta\varepsilon\zeta(\mathbf{s}\vartheta\eta){\cdot}\iota\beta\vartheta(\mathbf{s}\gamma\iota){<}\delta\eta)$. So we can consider $\boldsymbol{\beta}\alpha\beta\gamma\delta$ Gödel's β function relation of *RT* that is represented among the *T wffs*.

2.4.10. Definition of Relation Composition by Substitution in *T*

2.4.10.1. If α, β, γ, δ, ε, ζ, η, ϑ, … are *T wffs*, then let $(\mathbb{S}(\alpha\ldots\beta\ldots\zeta\ldots\vartheta\ldots\gamma)(\delta\ldots\beta)(\varepsilon\ldots\zeta)(\eta\ldots\vartheta)\ldots)$ be $\sum\beta\sum\zeta\sum\vartheta\ldots((\alpha\ldots\beta\ldots\zeta\ldots\vartheta\ldots\gamma)\wedge(\delta\ldots\beta)\wedge(\varepsilon\ldots\zeta)\wedge(\eta\ldots\vartheta)\wedge\ldots)$ (this one is the definition in *RT*) i.e. $\sum\beta\sum\zeta\sum\vartheta\ldots(D(D(\alpha\ldots\beta\ldots\zeta\ldots\vartheta\ldots\gamma)(\delta\ldots\beta)(\varepsilon\ldots\zeta)(\eta\ldots\vartheta)\ldots))$. So we can consider $(\mathbb{S}(\alpha\ldots\beta\ldots\zeta\ldots\vartheta\ldots\gamma)(\delta\ldots\beta)(\varepsilon\ldots\zeta)(\eta\ldots\vartheta)\ldots)$ the relation composition by substitution of *RT* that is represented among the *T wffs*.

2.4.11. Definition of Relation Composition by Recursion in *T*

2.4.11.1. If α, β, γ, δ, ε, ζ, η, ϑ, ι, κ, λ, μ, … are *T wffs*, then let $\mathbb{R}(\alpha\ldots\beta)(\gamma\ldots\delta\varepsilon\zeta)$ be $\sum\eta\sum\vartheta(D(D\sum\iota\sum\kappa(D(D\boldsymbol{\beta}\eta\vartheta\kappa\iota\mathbf{z}\kappa(\alpha\ldots\iota)\boldsymbol{\beta}\eta\vartheta\varepsilon\zeta\prod\iota(D{<}\iota\varepsilon\prod\kappa\prod\lambda\prod\mu(D\boldsymbol{\beta}\eta\vartheta\iota\kappa\boldsymbol{\beta}\eta\vartheta\mu\lambda\mathbf{s}\iota\mu(\gamma\ldots\iota\kappa\lambda)))))))$. This is the standard definition of relation composition by recursion in *RT*. So we can consider $\mathbb{R}(\alpha\ldots\beta)(\gamma\ldots\delta\varepsilon\zeta)$ the relation composition by recursion of *RT* that is represented among the *T wffs*.

2.4.12. Definition of Relation Composition by μ-operator in *T*

2.4.12.1. If α, β, γ, δ, … are *SSL wffs*, then let $\mu\beta(\alpha\ldots\beta\gamma)$ be $\sum\gamma(D(D\mathbf{z}\gamma(\alpha\ldots\beta\gamma)\prod\delta(D{<}\delta\beta(\alpha\ldots\delta\gamma)))$. This is the standard definition of relation composition by μ-operator in *RT*. So we can consider $\mu\beta(\alpha\ldots\beta\gamma)$ the relation composition by μ-operator of *RT* that is represented among the *T wffs*.

2.4.13. As projection relation (see 2.4.3), successor relation (see 2.4.4), zero relation (see 2.4.5), relation composition by substitution (see 2.4.10), relation composition by recursion (see 2.4.11) and relation composition by μ-operator (see 2.4.12) can be represented in *T*, all the recursive relations can be represented in *T* and so basic arithmetic too. Remember that *T* can prove its completeness and consistency in itself (see 2.3) and that it has got *SSL* as model (see 2.2). The consequence is that *SSL* can prove its completeness and consistency in itself and it can represent in itself all the recursive relations and the basic arithmetic. This perspective makes useless the distinction between general and primitive recursive functions.

3. CONCLUSION AND OUTLOOK

If the result of this paper is correct, then there are some important consequences both for Logic and for the emerging physics of information. We can restart Hilbert's original program, while Whitehead and Russell's approach becomes valid again, too. All logical and mathematical achievements that are linked to Gödel's theorems (Rosser, 1936) have to be revised. There would be very relevant implications in physics also, mainly as far as nanobiological systems are concernded. For instance, Salvatore Santoli has stressed strongly (Santoli, 2007) that Gödel's incompleteness theorems concern the physical reality very deeply in the case of dynamic hierarchical evolutionary systems. They are described by non-linear mathematics involving self-organization and an information flow through the hierarchy with the making of *abstractions* of always increasing rank, level by level, which describe the environment as *"simulations"* and *"compression"* through an always decreasing number of degrees of freedom (i.e., what is called *"abstraction"*) and with the creation of *meaning* from compression, i.e. of a semantics from dissipative processes in addition to mere syntax. If Gödel's theorems are true, then it is impossible to think of a minimal hierarchical level, i.e. the lowest irreducible cognitive level throughout the whole chain of levels of abstractions; or, stated otherwise, a ***self-consistent*** **and** ***complete*** axioms system. The hierarchy would be bottomless, a logically and physically unattainable depth. If these theorems are not valid, then there would be a hierarchical level of *"minimal abstraction"*, that might look like leading to paradoxes which actually could be shown illusory: indeed, the level structure in the logic space could be described by some corresponding physical processes soundly rooted in the phase space and in the ultimate physics of the Universe.

Anyway, it is the very basic intention of this paper to tackle the study of the questions concerning Gödel's theorems newly, as is impelled by the emergence of new fields of mathematics and physics, and suggested by recent criticisms: **let us start the debate!**

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