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Many-valued Sentence Logics: Representation Problems in Standard Sentence Logic

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1. Introduction

In this paper we study the representability of many-valued logics in standard sentence logic.

2. Disjunctive normal forms in Many-valued Logics

Let L be a many-valued logic at will with, for example, the n truth values $v_0,\ldots,v_{n-1}$. Select among $v_0,\ldots,v_{n-1}$ two truth values at will and call them v_i and v_j.

Define 'bel of v_i on v_j' (written '$\wedge_i^j$') every connective of L whose truth table is compatible with the following rules:

BEL 1: $(\forall k)\left('\, v_k \wedge_i^j v_i' = '\, v_i \wedge_i^j v_k' = '\, v_k'\right)$,

BEL 2: $(\forall k)\left('\, v_k \wedge_i^j v_j' = '\, v_j \wedge_i^j v_k' = '\, v_j'\right)$.

Define 'abo of v_i on v_j' (written '$\vee_i^j$') every connective of L whose truth table is compatible with the following rules:

ABO 1: $(\forall k)\left('\, v_k \vee_i^j v_i' = '\, v_i \vee_i^j v_k' = '\, v_i'\right)$,

ABO 2: $(\forall k)\left('\, v_k \vee_i^j v_j' = '\, v_j \vee_i^j v_k' = '\, v_k'\right)$.

Given any truth value v_k of L, define 'v_k-in of v_i on v_j' (written '$\left(v_k\right)_i^j$') the monadic connective whose truth table is compatible with the following rules:

IN 1: $'\left(v_k\right)_i^j v_k' = '\, v_i'$,

IN 2: $k \neq h \Leftrightarrow '\left(v_k\right)_i^j v_h' = '\, v_j'$.

Abbreviate '$\left(v_j\right)_i^j$' with $\sim_i^j$.

Given any truth value v_k of L, define 'v_k-out of v_i on v_j' (written '$\left[v_k\right]_i^j$') every monadic connective whose truth table is compatible with the following rules:

OUT 1: $'\left[v_k\right]_i^j v_i' = 'v_k'$,

OUT 2: $'\left[v_k\right]_i^j v_j' = 'v_j'$.

Define 'analogy of v_i on v_j' (written '$\equiv_i^j$') the connective of L whose truth table is compatible with the following rules:

ANL 1: $(\forall k)\left('v_k \equiv_i^j v_k' = 'v_i'\right)$,

ANL 2: $(\forall k)(\forall h)\left(k \neq h \Leftrightarrow 'v_k \equiv_i^j v_h' = 'v_j'\right)$.

Define 'absurdity of v_i on v_j' (written '$\varnothing_i^j$') any formula of L whose truth table contains values v_j only.

Consider now the formula $\varphi\left(p_1,\ldots,p_m\right)$ of L where $p_1,\ldots,p_m$ are atomic sentences. Suppose the following truth table:

$\varphi\left(p_1,\ldots,p_m\right)$	p_1	…	p_m
v_{o_1}	v_0	…	v_0
v_{o_2}	v_0	…	v_1
…	…	…	…
$v_{o_{n^m}}$	v_n	…	v_n

From the *n*-valued sentence calculus it is very easy to prove that $\varphi\left(p_1,\ldots,p_m\right)$ has the same truth function as:

$$\left[v_{o_1}\right]_i^j\left(\left(v_0\right)_i^j p_1 \wedge_i^j \ldots \wedge_i^j \left(v_0\right)_i^j p_m\right) \vee_i^j \left[v_{o_2}\right]_i^j\left(\left(v_0\right)_i^j p_1 \wedge_i^j \ldots \wedge_i^j \left(v_1\right)_i^j p_m\right) \vee_i^j$$

$$\vee_i^j \ldots \vee_i^j \left[v_{o_{n^m}}\right]_i^j\left(\left(v_n\right)_i^j p_1 \wedge_i^j \ldots \wedge_i^j \left(v_n\right)_i^j p_m\right)$$

We call the previous formula v_i, v_j – *disjunctive normal form* of the formula $\varphi\left(p_1,\ldots,p_m\right)$ in the considered logic.

Example 2.1

Consider the three-valued calculus in the most complete form. Consider also the following dyadic connective with its truth table:

pRq	*p*	*q*
1	2	2
0	2	1
0	2	0
2	1	2
1	1	1
2	1	0
2	0	2
1	0	1
0	0	0

The connectives that are definable, for example, as 'bel of 2 *on* 0 *' (written '* $\wedge_2^0$*')* *have the following rules in common:* $2 \wedge_2^0 2 = 2$, $2 \wedge_2^0 0 = 0$, $0 \wedge_2^0 2 = 0$, $0 \wedge_2^0 0 = 0$.

The connectives that are definable as 'abo of 2 *on* 0 *' (written '* $\vee_2^0$*')* *have the following rules in common:* $2 \vee_2^0 2 = 2$, $2 \vee_2^0 0 = 2$, $0 \vee_2^0 2 = 2$, $0 \vee_2^0 0 = 0$, $(k)(' v_k \vee_2^0 0' = ' 0 \vee_2^0 v_k ' = ' v_k ')$.

The connective ' 2*-in of* 2 *on* 0 *' (written '*$(2)_2^0$*')* *has the following truth table:* $(2)_2^0 2 = 2$, $2 \neq v_h \Leftrightarrow ' (2)_2^0 v_h ' = ' 0'$.

The connective ' 1*-in of* 2 *on* 0 *' (written '*$(1)_2^0$*')* *has the following truth table:* $(1)_2^0 1 = 2$, $1 \neq v_h \Leftrightarrow ' (1)_2^0 v_h ' = ' 0'$.

The connective ' 0*-in of* 2 *on* 0 *' (written '*$(0)_2^0$*')* *has the following truth table:* $(0)_2^0 0 = 2$, $0 \neq v_h \Leftrightarrow ' (0)_2^0 v_h ' = ' 0'$.

The connectives that are definable as ' 2*-out of* 2 *on* 0 *' (written '* $[2]_2^0$*')* *have the following rules in common:* $[2]_2^0 2 = 2$, $[2]_2^0 0 = 0$.

The connectives that are definable as '1-out of 2 on 0' (written '$[1]^0_2$') have the following rules in common: $[1]^0_2 2 = 1$, $[1]^0_2 0 = 0$.

The connectives that are definable as '0-out of 2 on 0' (written '$[0]^0_2$') have the following rules in common: $[0]^0_2 2 = 0$, $[0]^0_2 0 = 0$.

Finally, the connective 'analogy of 2 on 0' (written '$\equiv^0_2$') has: $(k)(' v_k \equiv^0_2 v_k ' = ' 2')$, $(k)(h)(k \neq h \Leftrightarrow ' v_k \equiv^0_2 v_h ' = ' 0')$.

The $2,0$-disjunctive normal form of pRq is the following:

$$[1]^0_2\left((2)^0_2 p \wedge^0_2 (2)^0_2 q\right) \vee^0_2 [0]^0_2\left((2)^0_2 p \wedge^0_2 (1)^0_2 q\right) \vee^0_2$$
$$\vee^0_2 [0]^0_2\left((2)^0_2 p \wedge^0_2 (0)^0_2 q\right) \vee^0_2 [2]^0_2\left((1)^0_2 p \wedge^0_2 (2)^0_2 q\right) \vee^0_2$$
$$\vee^0_2 [1]^0_2\left((1)^0_2 p \wedge^0_2 (1)^0_2 q\right) \vee^0_2 [2]^0_2\left((1)^0_2 p \wedge^0_2 (0)^0_2 q\right) \vee^0_2$$
$$\vee^0_2 [2]^0_2\left((0)^0_2 p \wedge^0_2 (2)^0_2 q\right) \vee^0_2 [1]^0_2\left((0)^0_2 p \wedge^0_2 (1)^0_2 q\right) \vee^0_2$$
$$\vee^0_2 [0]^0_2\left((0)^0_2 p \wedge^0_2 (0)^0_2 q\right)$$

The proof of truth function identity between the last formula and pRq by the three-valued calculus is left to the reader.

Example 2.2

Consider the standard sentence calculus and a dyadic connective at will, for example:

$p \div q$	p	q
0	1	1
1	1	0
1	0	1
0	0	0

The connective 'bel of 0 on 1' (written '$\wedge^1_0$') has the following truth table: $0 \wedge^1_0 0 = 0$, $0 \wedge^1_0 1 = 1$, $1 \wedge^1_0 0 = 1$, $1 \wedge^1_0 1 = 1$.

The connective 'abo of 0 on 1' (written '$\vee^1_0$') has the following truth table: $0 \vee^1_0 0 = 0$, $0 \vee^1_0 1 = 0$, $1 \vee^1_0 0 = 0$, $1 \vee^1_0 1 = 1$.

The connective ' 1-*in of* 0 *on* 1 ' *(written* '$(1)^1_0$') *has the following truth table:* $(1)^1_0 1 = 0$, $(1)^1_0 0 = 1$.

The connective ' 0-*in of* 0 *on* 1 ' *(written* '$(0)^1_0$') *has the following truth table:* $(0)^1_0 1 = 1$, $(0)^1_0 0 = 0$.

The connective ' 1-*out of* 0 *on* 1 ' *(written* '$[1]^1_0$') *has the following truth table:* $[1]^1_0 1 = 1$, $[1]^1_0 0 = 1$.

The connective ' 0-*out of* 0 *on* 1 ' *(written* '$[0]^1_0$') *has the following truth table:* $[0]^1_0 0 = 0$, $[0]^1_0 1 = 1$.

Finally, the connective '*analogy of* 0 *on* 1 ' *(written* '$\equiv^1_0$') *has the following truth table:* $0 \equiv^1_0 0 = 0$, $0 \equiv^1_0 1 = 1$, $1 \equiv^1_0 0 = 1$, $1 \equiv^1_0 1 = 0$.

Now, we can make a mapping from the last connectives and the standard connectives by truth table identity. We have:

Previous Connectives	*Standard Connectives*
$\wedge^1_0$	$\vee$
$\vee^1_0$	$\wedge$
$(1)^1_0$,	$\sim$
$(0)^1_0$, $[0]^1_0$	*monadic identity*
$[1]^1_0$	*monadic contradiction*

The previous mapping permits us to identify $0,1$ *– disjunctive normal form of* $p \div q$ *, i. e:*

$$[0]^1_0\left((1)^1_0 p \wedge^1_0 (1)^1_0 q\right) \vee^1_0 [1]^1_0\left((1)^1_0 p \wedge^1_0 (0)^1_0 q\right) \vee^1_0$$
$$\vee^1_0 [1]^1_0\left((0)^1_0 p \wedge^1_0 (1)^1_0 q\right) \vee^1_0 [0]^1_0\left((0)^1_0 p \wedge^1_0 (0)^1_0 q\right)$$

with the standard conjunctive normal form of $p \div q$*:*

$$(\sim p \vee \sim q) \wedge (p \vee q).$$

We shall let the reader prove the identity betwen the standard disjunctive normal form of $p \div q$ *and its* $1,0$ *– disjunctive normal form that use* $\wedge^0_1$, $\vee^0_1$, $(1)^0_1$, $(0)^0_1$, $[0]^0_1$, $[1]^0_1$ *instead of* $\wedge^1_0$, $\vee^1_0$, $(1)^1_0$, $(0)^1_0$, $[0]^1_0$, $[1]^1_0$.

We call v_i, v_j – *disjunct* every formula that has the form $\left[v_y\right]_i^j\left(\left(v_{x_1}\right)_i^j p_1 \wedge_i^j \ldots \wedge_i^j \left(v_{x_m}\right)_i^j p_m\right)$. It is very easy to prove that, given a logic L that has v_i, v_j among its truth values, every its formula has an equivalent v_i, v_j – disjunctive normal form, therefore, to study L, we can consider the v_i, v_j – disjunctive normal form only instead of all the possible formulas.

We shall let the reader prove the distributivity of a connective $\left[v_y\right]_i^j$, i. e. that $\left[v_y\right]_i^j\left(\left(v_{x_1}\right)_i^j p_1 \wedge_i^j \ldots \wedge_i^j \left(v_{x_m}\right)_i^j p_m\right)$ is equivalent to $\left[v_y\right]_i^j\left(v_{x_1}\right)_i^j p_1 \wedge_i^j \ldots \wedge_i^j \left[v_y\right]_i^j\left(v_{x_m}\right)_i^j p_m$.

3. Representation of Many-valued Logics in Standard Sentence Logic by Standard Predicative Calculus

Given any many-valued logic L with truth values v_i and v_j, it is possible to make a mapping of it in the standard predicative calculus also in the following way:

Many-valued Logic	Standard Predicative Calculus
$\left[v_y\right]_i^j\left(v_k\right)_i^j p_h$	$A_y^2\left(a_k, a_h\right)$
$\wedge_i^j$	$\wedge$
$\vee_i^j$	$\wedge$
$\varnothing_i^j$	A_j^0

Observe that to replace $\wedge_i^j$ and $\vee_i^j$ with the same standard connective $\wedge$ does not create mapping ambiguity because the sole indexes of $A_y^2\left(a_k, a_h\right)$ always permit reconstruction of the original disjunctive normal form.

Example 3.1

In example 2.1, the disjunctive normal form of pRq is:

$$[1]_2^0\left((2)_2^0 p \wedge_2^0 (2)_2^0 q\right) \vee_2^0 [0]_2^0\left((2)_2^0 p \wedge_2^0 (1)_2^0 q\right) \vee_2^0$$
$$\vee_2^0 [0]_2^0\left((2)_2^0 p \wedge_2^0 (0)_2^0 q\right) \vee_2^0 [2]_2^0\left((1)_2^0 p \wedge_2^0 (2)_2^0 q\right) \vee_2^0$$
$$\vee_2^0 [1]_2^0\left((1)_2^0 p \wedge_2^0 (1)_2^0 q\right) \vee_2^0 [2]_2^0\left((1)_2^0 p \wedge_2^0 (0)_2^0 q\right) \vee_2^0$$
$$\vee_2^0 [2]_2^0\left((0)_2^0 p \wedge_2^0 (2)_2^0 q\right) \vee_2^0 [1]_2^0\left((0)_2^0 p \wedge_2^0 (1)_2^0 q\right) \vee_2^0$$
$$\vee_2^0 [0]_2^0\left((0)_2^0 p \wedge_2^0 (0)_2^0 q\right)$$

If we put p equal to p_1 and q equal to p_2 and apply the distributivity of the connectives $[v_h]_j^i$ in the disjancts, our formula becomes :

$$[1]_2^0(2)_2^0 p_1 \wedge_2^0 [1]_2^0(2)_2^0 p_2 \vee_2^0 [0]_2^0(2)_2^0 p_1 \wedge_2^0 [0]_2^0(1)_2^0 p_2 \vee_2^0$$
$$\vee_2^0 [0]_2^0(2)_2^0 p_1 \wedge_2^0 [0]_2^0(0)_2^0 p_2 \vee_2^0 [2]_2^0(1)_2^0 p_1 \wedge_2^0 [2]_2^0(2)_2^0 p_2 \vee_2^0$$
$$\vee_2^0 [1]_2^0(1)_2^0 p_1 \wedge_2^0 [1]_2^0(1)_2^0 p_2 \vee_2^0 [2]_2^0(1)_2^0 p_1 \wedge_2^0 [2]_2^0(0)_2^0 p_2 \vee_2^0$$
$$\vee_2^0 [2]_2^0(0)_2^0 p_1 \wedge_2^0 [2]_2^0(2)_2^0 p_2 \vee_2^0 [1]_2^0(0)_2^0 p_1 \wedge_2^0 [1]_2^0(1)_2^0 p_2 \vee_2^0$$
$$\vee_2^0 [0]_2^0(0)_2^0 p_1 \wedge_2^0 [0]_2^0(0)_2^0 p_2$$

this last formula corresponds in standard predicative calculus to the formula:

$$A_1^2(a_2,a_1) \wedge A_1^2(a_2,a_2) \wedge A_0^2(a_2,a_1) \wedge A_0^2(a_1,a_2) \wedge$$
$$\wedge A_0^2(a_2,a_1) \wedge A_0^2(a_0,a_2) \wedge A_2^2(a_1,a_1) \wedge A_2^2(a_2,a_2) \wedge$$
$$\wedge A_1^2(a_1,a_1) \wedge A_1^2(a_1,a_2) \wedge A_2^2(a_1,a_1) \wedge A_2^2(a_0,a_2) \wedge$$
$$\wedge A_2^2(a_0,a_1) \wedge A_2^2(a_2,a_2) \wedge A_1^2(a_0,a_1) \wedge A_1^2(a_1,a_2) \wedge$$
$$\wedge A_0^2(a_0,a_1) \wedge A_0^2(a_0,a_2)$$

Finally, as regards as the theorems in the many-valued logics, the most general supposition is that if $v_0,\ldots,v_m,\ldots,v_{n-1}$ are the truth values of any many-valued logic, then the truth functions of corresponding theorems can contain only the truth values $v_0,\ldots,v_m$. Such a supposition can easily be represented in the standard predicative calculus by substitution of any formula $A_k^2(a_i,a_j)$, whose predicate corresponds to a truth value that can belong to theorem truth functions, with $\left(A_k^2(a_i,a_j) \supset A_k^2(a_i,a_j)\right)$.

Example 3.2

Given example 3.1, let as say that the theorem truth functions can contain the truth value 2 *only, so our formula becomes:*

$$A_1^2(a_2,a_1) \wedge A_1^2(a_2,a_2) \wedge A_0^2(a_2,a_1) \wedge A_0^2(a_1,a_2) \wedge A_0^2(a_2,a_1) \wedge$$
$$\wedge A_0^2(a_0,a_2) \wedge \left(A_2^2(a_1,a_1) \supset A_2^2(a_1,a_1)\right) \wedge \left(A_2^2(a_2,a_2) \supset A_2^2(a_2,a_2)\right) \wedge$$
$$\wedge A_1^2(a_1,a_1) \wedge A_1^2(a_1,a_2) \wedge \left(A_2^2(a_1,a_1) \supset A_2^2(a_1,a_1)\right) \wedge$$
$$\wedge \left(A_2^2(a_2,a_2) \supset A_2^2(a_2,a_2)\right) \wedge A_1^2(a_0,a_1) \wedge A_1^2(a_1,a_2) \wedge$$
$$\wedge A_0^2(a_0,a_1) \wedge A_0^2(a_0,a_2)$$

Observe that the sentences of standard first order predicative calculus, that represent the sentences of any many-valued logic in this paper, do not have variables and quantifiers, therefore they are also sentences of standard sentence logic and, from our construction, such formulas are two-valued tautologies iff they represent theorems of the being considered many-valued logic.

Other, and even better, representations of many-valued logics in the standard sentence logic with conservation of the theorems can exist, also better, but, in this paper, it is important to prove the existence of at least one of such representations.

Finally observe that in the present representation every formula of any many-valued logic is representable in standard sentence logic but not vice versa, this fact suggests that any many-valued logic is not an expansion of the two-valued logic, but could be a restriction instead. Other studies are necessary on such topics.

4. References

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