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Sentence Calculus and Fractals

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1. Introduction

In this work we show that it is possible a sentence calculus which is built on natural numbers and that every sentence connective corresponds to a numeric recursive function. In particular Sheffer's connective, which can define the remaining connectives, corresponds to a particular numeric recursive function which can define the corresponding numeric recursive functions. Such a function has some fractal properties.

2. Standard Sentence Calculus

The following propositions on standard sentence calculus are well known:

1) *There is a standard method to fixe the truth function of the atomic sentences.*

In fact, given a sentence $\mathcal{A}$ the truth fuction of which has to be calculated, call a_1 the first atomic sentence which appears in $\mathcal{A}$, a_2 the first atomic sentence which appears in $\mathcal{A}$ and which is not equal to a_1, a_3 the first atomic sentence which appears in $\mathcal{A}$ and which is equal neither to a_1 nor to a_2, and so on. Put the symbol *1* as true and the symbol *0* as false. Standardly, if $\mathcal{A}$ contains one distinct atomic sentence a_1, then the truth function of a_1 is $\begin{matrix} 1 \\ 0 \end{matrix}$, if $\mathcal{A}$ contains two distinct atomic sentences a_1 and a_2, then the truth function of a_1 is $\begin{matrix} 1 \\ 1 \\ 0 \\ 0 \end{matrix}$ and of a_2 is $\begin{matrix} 1 \\ 0 \\ 1 \\ 0 \end{matrix}$, if $\mathcal{A}$ contains three distinct atomic sentences a_1 , a_2 and a_3 , then the truth function

of a_1 is $\begin{matrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix}$, of a_2 is $\begin{matrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{matrix}$, of a_2 is $\begin{matrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{matrix}$, and so on.

2) *All the connectives of the standard sentence calculus are definable by the only Sheffer's 2-adic connective "|" which is defined by the following truth table:*

a_1	a_2	$a_1 \mid a_2$
1	*1*	*0*
1	*0*	*1*
0	*1*	*1*
0	*0*	*1*

3) *Therefore, all the sentence calculus is reducible to operate of the Sheffer's connective on the truth functions.*

3. From Standard Sentence Calculus to an Equivalent Numeric Sentence Calculus

Every truth function is a column of elements *1* and/or *0*. Therefore its matricial transposition is a row of elements *1* and/or *0*. So, if we replace *1* (i. e. true) with the digit 1 (i. e. one) and *0* (i. e. false) with the digit 0 (i. e. zero) in the matricial transposition of a truth function, then we obtain a binary number. e. g. we have:

truth function: $\begin{matrix} 1 \\ 1 \\ 0 \\ 0 \end{matrix}$;

its matricial transposition: *1100* ;
a corresponding binay number: 1100.

The correspondence between truth functions and binary numbers is not one-to-one because, e. g., the truth functions $\begin{matrix} \mathit{1} \\ \mathit{1} \end{matrix}$ and $\begin{matrix} \mathit{0} \\ \mathit{0} \\ \mathit{1} \\ \mathit{1} \end{matrix}$ correspond both to the binary number 11. In fact the digits 0 before the first digit 1 have not value in the binary numbers, but the truth values *0* over the truth values *1* have a great value in the truth functions!

So, to obtain an one-to-one mapping between truth functions and numeric expressions we must conserve in some way the information of the truth function lenght in the numeric expression. A way can be put in brackets the number of elements of a truth function behind the corresponding binary number. As the number of elements of a truth function of a sentence is the power of two of the number of the distinct atomic sentences which are contained in the given sentence, we can bracket this number of distinct atomic sentences instead of the number of elements of the corresponding truth functions. So, e. g., we have:

truth function: $\begin{matrix} \mathit{0} \\ \mathit{0} \\ \mathit{0} \\ \mathit{1} \end{matrix}$

corresponding numeric expression: 1(2);

truth function: $\begin{matrix} \mathit{0} \\ \mathit{1} \end{matrix}$

corresponding numeric expression: 1(1);

In this way, the desired one-to-one mapping is obtained. A Easy following step is the substitution of the binary numbers with the corresponding decimal numbers. So, the set of sentence truth functions corresponds one-to-one to a subset of the set of the natural numbers { 0, 1, 2, 3, … }.

An important consequence of this one-to-one mapping is that every sentence connective defines a numeric function in the natural numbers. As all the sentence connectives are definable by the Sheffer's connective, it is sufficient to study its corresponding numeric function which we call Sh (). In fact, all the other numeric functions which correspond to other connectives are definable in terms of Sh () because every connective is definable in terms of Sheffer's connective.

It is evident that the result of Sh () depends on the two numeric expression $x\ (\ n\ )$ and $y\ (\ n\ )$ which correspond to the truth functions of the two arguments of the Sheffer's connective. Therefore we can write Sh () = Sh (x (n), y (n)). But it is opportune consider also the particular cases in which n is irrelevant for the result of Sh ( ) $\left(\text{e. g. the case } \lim_{n \varnothing \bullet} n \right)$.
Therefore we put in Sh () also a term m which is equal to 0 in the case in which n is irrelevant and is greater than 0 in the case in which n is relevant. Therefore we can write Sh () = Sh (x (n), y (n), m). We can also abbreviate Sh (x (n), y (n), m) in Sh (x, y, m, n) where we call x 'first truth function', y 'second truth function', m 'cyclicity', n 'diameter'.
So, the standard sentence calculus is entirely representable in the formal number theory by the set of natural number { 0, 1, … } and by the numeric function Sh (x, y, m, n). Now we can find a way to calculate the function Sh (x, y, m, n) given x, y, m, n. We can do it by building of Sh (x, y, m, n) as recursive function.

4. Recursive Functions in General

Consider the following known set of primitive recursive function:

$$Z(x) = 0$$

zero function

$$N(x) = x + 1$$

successor function

$$U_i^n\left(x_1, \ldots, x_i, \ldots, x_n\right) = x_i$$

projection function

It is known that we can otain any recursive function by the following composition rules:

$$f\left(x_1, \ldots, x_n\right) = g\left(h_1\left(x_1, \ldots, x_n\right), \ldots, h_m\left(x_1, \ldots, x_n\right)\right)$$

substitution

$$\begin{cases} \mathrm{f}\left(x_1, \ldots, x_n , 0 \right) = \mathrm{g}\left(x_1, \ldots, x_n \right) \\ \mathrm{f}\left(x_1, \ldots, x_n , \mathrm{N}(y) \right) = \mathrm{h}\left(x_1, \ldots, x_n , y, \ \mathrm{f}\left(x_1, \ldots, x_n , y \right) \right) \end{cases}$$

recursion

$$\mathrm{f}\left(x_1, \ldots, x_n \right) = \mu \mathrm{y} \left(\mathrm{g}\left(x_1, \ldots, x_n , y \right) = 0 \right) \qquad \mu\text{-operator.}$$

E. g., we can obtain the following numeric recursive functions:

$$\begin{cases} \partial(0) = 0 \\ \partial(\mathrm{N}(y)) = y \end{cases} \qquad \begin{cases} x + 0 = \mathrm{U}_1^1(x) \\ x + \mathrm{N}(y) = \mathrm{N}(x + y) \end{cases} \qquad \begin{cases} x \cdot 0 = \mathrm{Z}(x) \\ x \cdot \mathrm{N}(y) = x \cdot y + x \end{cases}$$

$$\begin{cases} x^0 = \mathrm{N}(0) \\ x^{\mathrm{N}(y)} = x^y \cdot x \end{cases} \qquad \begin{cases} x \dot{-} 0 = \mathrm{U}_1^1(x) \\ x \dot{-} \mathrm{N}(y) = \partial(x \dot{-} y) \end{cases} \qquad |x - y| = (x \dot{-} y) + (y \dot{-} x)$$

$$\begin{cases} \mathrm{sg}(0) = 0 \\ \mathrm{sg}(\mathrm{N}(y)) = \mathrm{N}(0) \end{cases}$$

$$\begin{cases} \underline{\mathrm{sg}}(0) = \mathrm{N}(0) \\ \underline{\mathrm{sg}}(\mathrm{N}(y)) = 0 \end{cases} \qquad \begin{cases} 0! = \mathrm{N}(0) \\ (\mathrm{N}(y))! = y! \cdot \mathrm{N}(y) \end{cases}$$

$$\min(x, y) = x \dot{-} (x \dot{-} y) \qquad \max(x, y) = x + (x \dot{-} y)$$

$$\begin{cases} \mathrm{re}(x, 0) = 0 \\ \mathrm{re}(x, \mathrm{N}(y)) = \mathrm{N}(\mathrm{re}(x, y)) \cdot \mathrm{sg}\left(\left| \mathrm{N}(\mathrm{re}(x, y)) \right| \right) \end{cases}$$

$$\begin{cases} \mathrm{qu}(x, 0) = 0 \\ \mathrm{qu}(x, \mathrm{N}(y)) = \mathrm{qu}(x, y) \cdot \underline{\mathrm{sg}}\left(\left| \mathrm{U}_1^1(x) - \mathrm{N}(\mathrm{re}(x, y)) \right| \right) \end{cases}$$

5. Sh (*x* , *y* , *m* , *n*) as Recursive Function

Observe that the greatest number which can be result of the function Sh (x, y, m, n) is the result of the expression $2^{\left(2^n\right)} - 1$, i. e. the number corresponding to a truth function of a tautology where there are n distinct atomic sentences. In general, Sh (x, y, *m*, *n*) is equal to an expression $2^{\left(2^{n}\right)} - k$ where *k* belongs to the natural number set $\left\{0, 1, \ldots, 2^{\left(2^{n}\right)}\right\}$ and *k* depends only on *x* and *y* because the results of the Sheffer's connective depend only on the truth functions which correspond to the numbers *x* and *y*.

Also, for definition, *m* is relevant only for its equality or inequality to zero which makes relevant or irrelevant *n* for the result of Sh (*x*, *y*, *m*, *n*).
From the previous considerations we can put:

$$\mathrm{Sh}\,(x, y, m, n) = \left|\, \mathrm{sg}\,(m) \cdot 2^{\left(2^{n}\right)} - \mathrm{TD}\,(x, y) \,\right|$$

where the numeric functions sg(*x*), x^{y} and $|\, x - y \,|$ are defined previously and TD(*x*, *y*) is the numeric function which gives *k* as result. Now, we must study TD(*x*, *y*).

6. Study of the Function TD(*x*, *y*) and its Recursive Definition

By empirical study of the table of TD(*x*, *y*) we obtain the following result:

y / *x*	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	…
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	…
1	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	…
2	1	1	3	3	1	1	3	3	1	1	3	3	1	1	3	3	…
3	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4	…
4	1	1	1	1	5	5	5	5	1	1	1	1	5	5	5	5	…
5	1	2	1	2	5	6	5	6	1	2	1	2	5	6	5	6	…
6	1	1	3	3	5	5	7	7	1	1	3	3	5	5	7	7	…

7	1	2	3	4	5	6	7	8	1	2	3	4	5	6	7	8	…
8	1	1	1	1	1	1	1	1	9	9	9	9	9	9	9	9	…
9	1	2	1	2	1	2	1	2	9	10	9	10	9	10	9	10	…
10	1	1	3	3	1	1	3	3	9	9	11	11	9	9	11	11	…
11	1	2	3	4	1	2	3	4	9	10	11	12	9	10	11	12	…
12	1	1	1	1	5	5	5	5	9	9	9	9	13	13	13	13	…
13	1	2	1	2	5	6	5	6	9	10	9	10	13	14	13	14	…
14	1	1	3	3	5	5	7	7	9	9	11	11	13	13	15	15	…
15	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	…
…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…

By the regular identity of some squares which are individualized by the internal borders of the previous table it is very easy to define TD (x, y) recursively. A way can be the following:

$$\begin{cases} \mathrm{MP2}(0) = \mathrm{N}(0) \\ \mathrm{MP2}(\mathrm{N}(y)) = \mathrm{MP2}\Big(\mathrm{qu}\big(y, \mathrm{N}(\mathrm{N}(0))\big)\Big) \cdot \mathrm{N}(\mathrm{N}(0)) \end{cases}$$

$$\mathrm{Qua}(x, y) = \mathrm{N}(0) - \mathrm{Sg}\Big(\mathrm{qu}\big(\max(x, y), \mathrm{MP2}(\min(x, y))\big)\Big)$$

$$\mathrm{Ter}(x, y) = \mathrm{qu}\big(\mathrm{MP2}(\min(x, y)), \mathrm{N}(0) + \mathrm{Qua}(x, y)\big)$$

$$\begin{cases} \mathrm{TD}(0, 0) = \mathrm{N}(0) \\ \mathrm{TD}(x, y) = \mathrm{TD}\big(\mathrm{re}(x, \mathrm{Ter}(x, y)), \mathrm{re}(y, \mathrm{Ter}(x, y))\big) + \mathrm{Ter}(x, y) \cdot \mathrm{Qua}(x, y) \end{cases}$$

So Sh(x, y, m, n) is defined as recursive function entirely and therefore it can be easily calculated.

7. Sh(*x*, *y*, *m*, *n*) as Primitive Recursive Function

It is known that Sheffer's connective x | y can define any other connective in standard sentence logic. Similarly, we observe that also the corresponding recursive function Sh(x, y, m, n) can define any other recursive function by the rules of substitution, recursion, and μ-operator. To prove this fact it is sufficient to define the three primitive recursive functions $Z(x) = 0$ (zero function), $N(x) = x + 1$ (successor function) and

$U_i^n(x_1, \ldots, x_i, \ldots, x_n) = x_i$ (projection function) in terms of Sh(x, y, m, n) by the rules of substitution, recursion and μ-operator. The wanted definitions are, e. g., the following:

$$N(x) = Sh(x, x, 0, x)$$

$$Z(x) = Sh(Sh(0,0,0,x), Sh(0,0,0,x), Sh(0,0,0,x), 0)$$

$$\begin{cases} \partial(0) = 0 \\ \partial(N(x)) = x \end{cases}$$

$$U_1^2(x_1, x_2) = Sh(\partial(x_1), \partial(x_1), 0, x_2)$$

$$U_2^2(x_1, x_2) = U_1^2(x_2, x_1)$$

$$U_i^n(x_1, \ldots, x_i, \ldots, x_n) = U_2^2(x_1\, U_2^2(x_2 \ldots U_2^2(x_{i-1}\, U_1^2(x_i\, U_1^2(x_{i+1} \ldots U_1^2(x_{n-1}\, x_n))))))$$

8. Fractal Properties of Sh (*x* , *y*, *m* , *n*)

The empirical study of the table of TD(x, y), component function of Sh (x , y, m , n), permitts us an other observation.
Infact the regular identity of some squares which are individualized by the internal borders of the table of TD(x, y), that we have previously shown, conforms itself to a typical fractal property: a part repeats the whole. This result needs new studies.

9. References

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