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# On the Time Logic

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## 1. Introduction

In this work, to show that we can build a 2-valued time logic which is valid also in indeterministic worlds, we propose a formal structure for the time logic and we study its relations with some Łoś' and Prior's results.

## 2. A Formal Integer Number Theory

In this section we define a formal structure **S** which can represent the non-negative integer number theory.

Consider the following set of symbols: 0, 1,…, =, (, ), ∀, ∃, ß, π, +, ·, -, $y_1$, …, $y_n$, …, ¬, ∧, ∨, ⊃, ≡.

Call 0 the individual constant zero. Call "=" 2- adic identity predicate (read it "… is equal to …"). Call "ß" 1-adic successor function (read it "the successor of …"). Call "π" 1-adic predecessor function (read it "the predecessor of …").Call "+" 2-adic sum function (read it "… plus …") Call "·" 2-adic product function (read it "… multiplied by …"). Call $x_n$ variable. Call ¬ 1-adic connective *not*, ∧ 2-adic connective *and*, ∨ 2-adic connective *inclusive or*, ⊃ 2-adic connective *implication*, ≡ 2-adic connective *equivalence*.

Put that variables and 0 are *terms*. Put that if $\tau_1$, $\tau_2$ are terms, then $ß\tau_1$, $\pi\tau_1$, $\tau_1 + \tau_2$, $\tau_1 \cdot \tau_2$ are terms.

Put that if $\tau_1$, $\tau_2$ are terms, then call $\tau_1 = \tau_2$ *atomic sententence*. If $\mathcal{A}$ is an atomic sentence, then $\mathcal{A}$ and $\neg\mathcal{A}$ are *literals*. Put that every literal is a sentence. Put that if $\mathcal{A}$ and $\mathcal{B}$ are sentences, then $\neg\mathcal{A}$, $\mathcal{A}\wedge\mathcal{B}$, $\mathcal{A}\vee\mathcal{B}$, $\mathcal{A}\supset\mathcal{B}$, $\mathcal{A}\equiv\mathcal{B}$ are sentences.

Call $(\forall y_n)$ *universal quantifier*. Call $(\exists y_n)$ *existential quantifier*. Call the universal and existential quantifiers *quantifiers*. Call $y_n$ in $(\forall y_n)$ and $(\exists y_n)$ *variable of the quantifier*. If $(Qy_n)$ is a quantifier with variable $y_n$ and $\mathcal{A}$ is a sentence, then $(Qy_n)\mathcal{A}$ is a sentence where we call $\mathcal{A}$ *scope* of the quantifier $(Qy_n)$.

There are not other ways to build a sentence of S.

Put that an occurrence of a variable $y_n$ in a sentence $\mathcal{A}$ is *linked* iff it is in the scope of a quantifier which has the same variable. Put that an occurrence of a variable $y_n$ in a sentence $\mathcal{A}$ is *free* iff it is not linked. Call the term $\tau_n$ *free for* $y_i$ *in* $\mathcal{A}$ iff no occurrence of $y_i$ in $\mathcal{A}$ is in the scope of a quantifier whose variable is in $\tau_n$.

Put the following abbreviations where $n$ and $m$ are natural numbers. Let:

| | | |
|---|---|---|
| $\mathcal{A}\wedge\mathcal{B}$ | be | $\neg(\mathcal{A}\supset\neg\mathcal{B})$, |
| $\mathcal{A}\vee\mathcal{B}$ | be | $\neg\mathcal{A}\supset\mathcal{B}$, |
| $\mathcal{A}\equiv\mathcal{B}$ | be | $\neg((\mathcal{A}\supset\mathcal{B})\supset\neg(\mathcal{B}\supset\mathcal{A}))$, |
| $(\exists y_n)\mathcal{A}$ | be | $\neg(\forall y_n)\neg\mathcal{A}$, |
| $y_n < y_m$ | be | $(\exists y_p)(\neg y_p=0\wedge y_p+y_n=y_m)$, |
| $n$ | be | ß … $n$ times … ß 0, |
| $-n$ | be | $\pi$ … $n$ times … $\pi$ 0. |
| $m - n$ | be | $m + (-n)$ |

Put the following axiom outlines:

A1: $\mathcal{A}\supset(\mathcal{B}\supset\mathcal{A})$,

A2: $(\mathcal{A}\supset(\mathcal{B}\supset\mathcal{C}))\supset((\mathcal{A}\supset\mathcal{B})\supset(\mathcal{A}\supset\mathcal{C}))$,

A3: $(\neg\mathcal{B}\supset\neg\mathcal{A})\supset((\neg\mathcal{B}\supset\mathcal{A})\supset\mathcal{B})$,

A4: If $\tau$ is a term free for $y_{\mathrm{i}}$ in $\mathcal{A}(y_n)$, then $(\forall y_n)\mathcal{A}(y_n)\supset\mathcal{A}(\tau)$,

A5: If $\mathcal{A}$ does not contain free occurrences of $y_{\mathrm{i}}$, then $(\forall y_n)(\mathcal{A}\supset\mathcal{B})\supset(\mathcal{A}\supset(\forall y_n)\mathcal{B})$.

Put the following axioms:

S1: $y_1=y_1$,
S2: $y_1=y_2\supset y_2=y_1$,
S3: $y_1=y_2\supset(y_2=y_3\supset y_1=y_3)$,
S4: $y_1=y_2\supset ßy_2=ßy_1$,
S5: $y_1=y_2\supset(y_1+y_3=y_2+y_3\wedge y_3+y_1=y_3+y_2)$,
S6: $y_1=y_2\supset(y_1\cdot y_3=y_2\cdot y_3\wedge y_3\cdot y_1=y_3\cdot y_2)$,
S7: $ßy_1=ßy_2\supset y_2=y_1$,
S8: $y_1=ßy_2\equiv y_2=\pi y_1$
S9: $(\exists y_2)(y_1=ßy_2)$,
S10: $y_1+0=y_1$,
S11: $y_1+ßy_2=ß(y_1+y_2)$,
S12: $y_1+\pi y_2=\pi(y_1+y_2)$,
S13: $y_1\cdot 0=0$,
S14: $y_1\cdot ßy_2=ß(y_1\cdot y_2)+y_2$,
S15: $y_1\cdot\pi y_2=\pi(y_1\cdot y_2)+y_2$.

Put the following inference outlines:

MP: $\mathcal{B}$ is deducible from $\mathcal{A}$ and $\mathcal{A}\supset\mathcal{B}$,

Gen: $(\forall y_i)\mathcal{A}$ is deducible from $\mathcal{A}$.

The formal theory S is adequate to represent the integer number theory. We let the proof to the reader.

## 3. The formal theory V

Consider the ordered set of infinite integers such that the first $n$ places (where $n$ is a natural number) are occupied by integers which can be distinct among them and the last infinite places are occupied by the same integer. Put them as individual constants.

Put the variables $z_1$ , …, $z_n$ , … .

Let $A$, $B$, $C$, $D$, $E$, $F$ be individual constants.

If $A$ is equal to $m, \ldots, n, p, p, p, \ldots$ and $B$ is equal to $q, \ldots, r, s, s, s, \ldots$ , then:

$ßA$ is $ßm, \ldots, ßn, ßp, ßp, ßp, \ldots$ ;
$\pi A$ is $\pi m, \ldots, \pi n, \pi p, \pi p, \pi p, \ldots$ ;
$A + B$ is $m +q, \ldots, n +r, p +s , p +s , p +s , \ldots$ ;
$A - B$ is $m -q, \ldots, n -r, p -s , p -s , p -s , \ldots$ ;
$A \cdot B$ is $m \cdot q, \ldots, n \cdot r, p \cdot s , p \cdot s , p \cdot s , \ldots$ ;
$A, B$ is $m, \ldots, n, p, q, \ldots, r, s, s, s, \ldots$ ;
$(A)B$ is $A, B$ .
$A=B$ is $m =q, \ldots, n =r, p =s , p =s , p =s , \ldots$ .
$A<B$ is $(A=m,C \wedge B=n, D \wedge m<n)\vee(A=C,D \wedge D=E,F \wedge C=E \wedge D<F)$

The variables and the individual constants are terms. If $A$ and $B$ are terms, then $ßA$, $\pi A$, $A + B$, $A - B$, $A \cdot B$, $(A)B$ are terms.

If $A$ and $B$ are terms, then $A = B$ and $A < B$ are atomic sentences. If $\mathcal{A}$ is an atomic sentence, then $\mathcal{A}$ and $\neg\mathcal{A}$ are *litterals*.

Put that every litteral is a sentence. Put that if $\mathcal{A}$ and $\mathcal{B}$ are sentences, then $\neg\mathcal{A}$, $\mathcal{A}\wedge\mathcal{B}$, $\mathcal{A}\vee\mathcal{B}$, $\mathcal{A}\supset\mathcal{B}$, $\mathcal{A}\equiv\mathcal{B}$ are sentences.

Call $(\forall y_n)$ *universal quantifier*. Call $(\exists y_n)$ *existential quantifier*. Call the universal and existential quantifiers *quantifiers*. Call $y_n$ in $(\forall y_n)$ and $(\exists y_n)$ *variable of the quantifier*. If $(\mathrm{Q}y_n)$ is a quantifier with variable $y_n$ and $\mathcal{A}$ is a sentence, then $(\mathrm{Q}y_n)\mathcal{A}$ is a sentence where we call $\mathcal{A}$ *scope* of the quantifier $(\mathrm{Q}y_n)$.

There are not other ways to build a sentence of **V**.

Put that an occurrence of a variable $z_n$ in a sentence $\mathcal{A}$ is *linked* iff it is in the scope of a quantifier which has the same variable. Put that an occurrence of a variable $z_n$ in a sentence $\mathcal{A}$ is *free* iff it is not linked. Call the term $\tau_n$ *free for* $z_{\mathrm{i}}$ *in* $\mathcal{A}$ iff no occurrence of $y_{\mathrm{i}}$ in $\mathcal{A}$ is in the scope of a quantifier whose variable is in $\tau_n$.

Put the following abbreviations. Let:

| | | |
|---|---|---|
| $\mathcal{A}\wedge\mathcal{B}$ | be | $\neg(\mathcal{A}\supset\neg\mathcal{B})$, |
| $\mathcal{A}\vee\mathcal{B}$ | be | $\neg\mathcal{A}\supset\mathcal{B}$, |
| $\mathcal{A}\equiv\mathcal{B}$ | be | $\neg((\mathcal{A}\supset\mathcal{B})\supset\neg(\mathcal{B}\supset\mathcal{A}))$, |
| $(\exists y_n)\mathcal{A}$ | be | $\neg(\forall y_n)\neg\mathcal{A}$, |
| $z_n>z_m$ | be | $z_m<z_n$, |
| $z_n\leq z_m$ | be | $z_n=z_m \wedge z_n<z_m$, |
| $z_n\geq z_m$ | be | $z_n=z_m \wedge z_n>z_m$. |

Put the following axiom outlines:

A1: $\mathcal{A}\supset(\mathcal{B}\supset\mathcal{A})$,

A2: $(\mathcal{A}\supset(\mathcal{B}\supset\mathcal{C}))\supset((\mathcal{A}\supset\mathcal{B})\supset(\mathcal{A}\supset\mathcal{C}))$,

A3: $(\neg\mathcal{B}\supset\neg\mathcal{A})\supset((\neg\mathcal{B}\supset\mathcal{A})\supset\mathcal{B})$,

A4: If $\tau$ is a term free for $z_{\mathrm{i}}$ in $\mathcal{A}(z_n)$, then $(\forall z_n)\mathcal{A}(z_n)\supset\mathcal{A}(\tau)$,

A5: If $\mathcal{A}$ does not contain free occurrences of $z_n$, then $(\forall z_n)(\mathcal{A}\supset\mathcal{B})\supset(\mathcal{A}\supset(\forall z_n)\mathcal{B})$.

Let (0, …) be a set of infinite 0. Put the following axioms:

S1: $z_1=z_1$,
S2: $z_1=z_2\supset z_2=z_1$,
S3: $z_1=z_2\supset(z_2=z_3\supset z_1=z_3)$,
S4: $z_1=z_2\supset ß z_2=ß z_1$,
S5: $z_1=z_2\supset(z_1+z_3=z_2+z_3\wedge z_3+z_1=z_3+z_2)$,
S6: $z_1=z_2\supset(z_1\cdot z_3=z_2\cdot z_3\wedge z_3\cdot z_1=z_3\cdot z_2)$,
S7: $ß z_1=ß z_2\supset z_2=z_1$,
S8: $z_1=ß z_2\equiv z_2=\pi z_1$
S9: $(\exists z_2)(z_1=ß z_2)$,
S10: $z_1+0=z_1$,
S11: $z_1+ß z_2=ß(z_1+z_2)$,
S12: $z_1+\pi z_2=\pi(z_1+z_2)$,
S13: $z_1\cdot 0=0$,
S14: $z_1\cdot ß z_2=ß(z_1\cdot z_2)+z_2$,
S15: $z_1\cdot\pi z_2=\pi(z_1\cdot z_2)+z_2$.

Put the following inference outlines:

MP: $\mathcal{B}$ is deducible from $\mathcal{A}$ and $\mathcal{A}\supset\mathcal{B}$,

Gen: $(\forall z_i)\mathcal{A}$ is deducible from $\mathcal{A}$.

## 4. A Formal Theory for the First Order Predicative Calculus

In this section we define a formal structure **K** which can represent the first order predicative calculus.

Consider the following set of symbols where $A$ and $B$ are terms at will of **V** and $m$ and $n$ are natural numbers:

$$a_1, \ldots, a_n, \ldots, \overset{0}{A}_A, \ldots, \overset{m}{A}_B, \ldots, (,), \forall, \exists, f_1^1, \ldots, f_n^m, \ldots, x_1, \ldots, x_n, \ldots, \sim, \wedge, \vee, \supset, \equiv.$$

Call $a_n$ individual constant. Call $\overset{m}{A}_A$ $m$ -adic *predicative letters*. Call $\overset{0}{A}_A$ *sentential variables*. Call $f_n^m$ $m$ -adic functional letter. Call $x_n$ variable. Call $\neg$ 1-adic connective *negation*, $\wedge$ 2-adic connective *and*, $\vee$ 2-adic connective *inclusive or*, $\supset$ 2-adic connective *implication*, $\equiv$ 2-adic connective *equivalence*.

Put that $a_n$ and $x_n$ are *terms*. Put that if $\tau_1, \ldots, \tau_m$ are terms, then $f_n^m(\tau_1, \ldots, \tau_m)$ is a term.

Put $\tau_1, \ldots, \tau_m$ are terms. So $A_{A}^{m}(\tau_1, \ldots, \tau_m)$ is an *atomic sententence*. If $\mathcal{A}$ is an atomic sentence or a sentential variable, then $\mathcal{A}$ and $\neg\mathcal{A}$ are *literals*.

Put that every litteral is a sentence. Put that if $\mathcal{A}$ and $\mathcal{B}$ are sentences, then $\neg\mathcal{A}$, $\mathcal{A}\wedge\mathcal{B}$, $\mathcal{A}\vee\mathcal{B}$, $\mathcal{A}\supset\mathcal{B}$, $\mathcal{A}\equiv\mathcal{B}$ are sentences.

Call $(\forall x_n)$ *universal quantifier*. Call $(\exists x_n)$ *existential quantifier*. Call the universal and existential quantifiers *quantifiers*. Call $x_n$ in $(\forall x_n)$ and $(\exists x_n)$ *variable of the quantifier*. If $(Qx_n)$ is a quantifier with variable $x_n$ and $\mathcal{A}$ is a sentence, then $(Qx_n)\mathcal{A}$ is a sentence where we call $\mathcal{A}$ *scope* of the quantifier $(Qx_n)$.

There are not other ways to build a sentence of **K**.

Put that an occurrence of a variable $x_n$ in a sentence $\mathcal{A}$ is *linked* iff it is in the scope of a quantifier which has the same variable. Put that an occurrence of a variable $x_n$ in a sentence $\mathcal{A}$ is *free* iff it is not linked. Call the term $\tau_n$ *free for* $x_i$ *in* $\mathcal{A}$ iff no occurrence of $x_i$ in $\mathcal{A}$ is in the scope of a quantifier whose variable is in $\tau_n$.

Put the following abbreviations. Let:

| | | |
|---|---|---|
| $\mathcal{A}\wedge\mathcal{B}$ | be | $\neg(\mathcal{A}\supset\neg\mathcal{B})$, |
| $\mathcal{A}\vee\mathcal{B}$ | be | $\neg\mathcal{A}\supset\mathcal{B}$, |
| $\mathcal{A}\equiv\mathcal{B}$ | be | $\neg((\mathcal{A}\supset\mathcal{B})\supset\neg(\mathcal{B}\supset\mathcal{A}))$, |
| $(\exists x_n)\mathcal{A}$ | be | $\neg(\forall x_n)\neg\mathcal{A}$, |

Put the following axiom outlines:

A1: $\mathcal{A}\supset(\mathcal{B}\supset\mathcal{A})$,

A2: $(\mathcal{A}\supset(\mathcal{B}\supset\mathcal{C}))\supset((\mathcal{A}\supset\mathcal{B})\supset(\mathcal{A}\supset\mathcal{C}))$,

A3: $(\neg\mathcal{B}\supset\neg\mathcal{A})\supset((\neg\mathcal{B}\supset\mathcal{A})\supset\mathcal{B})$,

A4: If $\tau$ is a term free for $x_n$ in $\mathcal{A}(x_n)$, then $(\forall x_n)\mathcal{A}(x_n)\supset\mathcal{A}(\tau)$,

A5: If $\mathcal{A}$ does not contain free occurrences of $x_n$, then $(\forall x_n)(\mathcal{A}\supset\mathcal{B})\supset(\mathcal{A}\supset(\forall x_n)\mathcal{B})$.

Put the following inference outlines:

MP: $\mathcal{B}$ is deducible from $\mathcal{A}$ and $\mathcal{A}\supset\mathcal{B}$,

Gen: $(\forall x_i)\mathcal{A}$ is deducible from $\mathcal{A}$.

The formal theory **K** is adequate to represent the first order predicative calculus. We let the proof to the reader. See also Mendelson [Mendelson, 1964].

### 5. A Time Logic for K

Consider a formal structure **T** which we obtain in the follwing way. Put in **T** all the terms and the sentences of **K** and **V**. Put that they are theorems in T iff they are theorems in **K** and **V** respectively.

Put that if $\mathcal{A}$ and $\mathcal{B}$ are sentences, then $\neg\mathcal{A}$, $\mathcal{A}\wedge\mathcal{B}$, $\mathcal{A}\vee\mathcal{B}$, $\mathcal{A}\supset\mathcal{B}$, $\mathcal{A}\equiv\mathcal{B}$ are sentences.

Call $(\forall x_n)$ *universal quantifier*. Call $(\exists x_n)$ *existential quantifier*. Call the universal and existential quantifiers *quantifiers*. Call $x_n$ in $(\forall x_n)$ and $(\exists x_n)$ *variable of the quantifier*.

The following affirmations on $x_n$ are true also for $z_n$.

If $(Qx_n)$ is a quantifier with variable $x_n$ and $\mathcal{A}$ is a sentence, then $(Qx_n)\mathcal{A}$ is a sentence where we call $\mathcal{A}$ *scope* of the quantifier $(Qx_n)$.

Put that an occurrence of a variable $x_n$ in a sentence $\mathcal{A}$ is *linked* iff it is in the scope of a quantifier which has the same variable. Put that an occurrence of a variable $x_n$ in a sentence $\mathcal{A}$ is *free* iff it is not linked. Call the term $\tau_n$ *free for* $x_i$ *in* $\mathcal{A}$ iff no occurrence of $x_i$ in $\mathcal{A}$ is in the scope of a quantifier whose variable is in $\tau_n$.

Put the following abbreviations where $n$ and $m$ are non-negative integer numbers and $A$, $B$, $C$ and $D$ are terms of **V**. Let:

| | | |
|---|---|---|
| $(\overset{0}{A}_A)_B$ | be | $\overset{0}{A}_{(A)B}$, |
| $(\overset{m}{A}_A)_B$ | be | $\overset{m}{A}_{(A)B}$, |
| $(\overset{m}{A}_A(\tau_1, \ldots, \tau_m))_B$ | be | $\overset{m}{A}_{(A)B}(\tau_1, \ldots, \tau_m)$, |
| $((Qx_n)\mathcal{A})_A$ | be | $(Qx_n)(\mathcal{A}_A)$, |
| $(\neg\mathcal{A})_A$ | be | $\neg(\mathcal{A}_A)$, |
| $(\mathcal{A}\supset\mathcal{B})_A$ | be | $\mathcal{A}_A\supset\mathcal{B}_A$, |
| $(\mathcal{A}\wedge\mathcal{B})_A$ | be | $\mathcal{A}_A\wedge\mathcal{B}_A$, |
| $(\mathcal{A}\vee\mathcal{B})_A$ | be | $\mathcal{A}_A\vee\mathcal{B}_A$, |
| $(\mathcal{A}\equiv\mathcal{B})_A$ | be | $\mathcal{A}_A\equiv\mathcal{B}_A$, |
| $(\mathcal{A}_A)_B$ | be | $\mathcal{A}_{(A)B}$. |

Put the following abbreviations. Let:

| | | |
|---|---|---|
| $\mathcal{A}\wedge\mathcal{B}$ | be | $\neg(\mathcal{A}\supset\neg\mathcal{B})$, |
| $\mathcal{A}\vee\mathcal{B}$ | be | $\neg\mathcal{A}\supset\mathcal{B}$, |
| $\mathcal{A}\equiv\mathcal{B}$ | be | $\neg((\mathcal{A}\supset\mathcal{B})\supset\neg(\mathcal{B}\supset\mathcal{A}))$, |
| $(\exists x_n)\mathcal{A}$ | be | $\neg(\forall x_n)\neg\mathcal{A}$, |

Put the following abbreviations where $n$ is a natural number. Let:

| | | |
|---|---|---|
| $\prod\mathcal{A}$ | be | $\mathcal{A}$, |
| $\sum\mathcal{A}$ | be | $\neg\prod\neg\mathcal{A}$, |

Put the following abbreviations. Let:

| | | |
|---|---|---|
| $\mathcal{A}\leq\mathcal{B}$ | be | $(\mathcal{A}\equiv\mathcal{C}_A)\wedge(\mathcal{B}\equiv\mathcal{D}_B)\wedge(A\leq B)$, |
| $\mathcal{A}\geq\mathcal{B}$ | be | $(\mathcal{A}\equiv\mathcal{C}_A)\wedge(\mathcal{B}\equiv\mathcal{D}_B)\wedge(A\geq B)$, |
| $\mathcal{A}=\mathcal{B}$ | be | $(\mathcal{A}\equiv\mathcal{C}_A)\wedge(\mathcal{B}\equiv\mathcal{D}_B)\wedge(A=B)$, |
| $\mathcal{A}<\mathcal{B}$ | be | $(\mathcal{A}\equiv\mathcal{C}_A)\wedge(\mathcal{B}\equiv\mathcal{D}_B)\wedge(A<B)$, |
| $\mathcal{A}>\mathcal{B}$ | be | $(\mathcal{A}\equiv\mathcal{C}_A)\wedge(\mathcal{B}\equiv\mathcal{D}_B)\wedge(A>B)$, |
| $\mathcal{A}\Delta\mathcal{B}$ | be | $(\forall z_n)\left(\mathcal{A}_{z_n}\equiv\mathcal{B}_{z_n}\right)$, |

Put the following axiom outlines:

A1: $\mathcal{A}\supset(\mathcal{B}\supset\mathcal{A})$,

A2: $(\mathcal{A}\supset(\mathcal{B}\supset\mathcal{C}))\supset((\mathcal{A}\supset\mathcal{B})\supset(\mathcal{A}\supset\mathcal{C}))$,

A3: $(\neg\mathcal{B}\supset\neg\mathcal{A})\supset((\neg\mathcal{B}\supset\mathcal{A})\supset\mathcal{B})$,

A4: If $\tau$ is a term free for $x_n$ in $\mathcal{A}(x_n)$, then $(\forall x_n)\mathcal{A}(x_n)\supset\mathcal{A}(\tau)$,

A5: If $\mathcal{A}$ does not contain free occurrences of $x_n$, then $(\forall x_n)(\mathcal{A}\supset\mathcal{B})\supset(\mathcal{A}\supset(\forall x_n)\mathcal{B})$.

Put the following axiom outline:

T1: $\mathcal{A}\supset\mathcal{A}_B$

Put the following inference outlines:

MP: $\mathcal{B}$ is deducible from $\mathcal{A}$ and $\mathcal{A}\supset\mathcal{B}$,

Gen: $(\forall x_i)\mathcal{A}$ is deducible from $\mathcal{A}$.

Give the following semantic interpretations. Let:

| | | |
|---|---|---|
| $\prod\mathcal{A}$ | be | "$\mathcal{A}$ is true always", |
| $\sum\mathcal{A}$ | be | "$\mathcal{A}$ is true sometimes", |
| $\mathcal{A}\leq\mathcal{B}$ | be | "$\mathcal{A}$ is before of or contemporary to $\mathcal{B}$", |
| $\mathcal{A}\geq\mathcal{B}$ | be | "$\mathcal{A}$ is after of or contemporary to $\mathcal{B}$", |
| $\mathcal{A}=\mathcal{B}$ | be | "$\mathcal{A}$ is contemporary to $\mathcal{B}$", |
| $\mathcal{A}<\mathcal{B}$ | be | "$\mathcal{A}$ is before $\mathcal{B}$", |
| $\mathcal{A}>\mathcal{B}$ | be | "$\mathcal{A}$ is after $\mathcal{B}$", |
| $\mathcal{A}\Delta\mathcal{B}$ | be | "$\mathcal{A}$ during $\mathcal{B}$", |
| $\mathcal{A}_B$ | be | "$\mathcal{A}$ true in the time interval $B$". |

**4. The Łoś' Time Logic.**

J. Łoś built the first axiomatic system of the time logic in the year 1947. He considered a first order predicative calculus with identity to wich he added nine axioms and an inference outline. By using the standard sentential symbolism - see Pizzi [1974] - we can write these nine axioms in this way:

R1: $\mathrm{R}t\neg p \equiv \neg\mathrm{R}tp$,
R2: $\mathrm{R}t(p\supset q)\supset(\mathrm{R}tp\supset\mathrm{R}tq)$,
R3: $\mathrm{R}t(((p\supset q)\supset(q\supset r))\supset(p\supset q))$,
R4: $\mathrm{R}t(p\supset(\neg p\supset q))$,
R5: $\mathrm{R}t((\neg p\supset p)\supset p)$,
R6: $\forall t\mathrm{R}tp\supset p$,
R7: $\forall t\forall n\exists t'\forall p(\mathrm{R}(t+n)p\equiv\mathrm{R}t'p)$,
R8: $\forall t\forall n\exists t'\forall p(\mathrm{R}(t'+n)p\equiv\mathrm{R}tp)$,
R9: $\forall t\exists p\forall t'(\mathrm{R}t'p\equiv(\forall q)(\mathrm{R}tq\equiv\mathrm{R}t'q))$.

Obviously $p$ and $q$ have to be interpreted as sentential variables, R as a dyadic operator which can be read "is satisfied at", " is true at", "is realized at", $t$ and $t'$ as terms of **V**.

To represent the Łoś' time logic in **T**, put the following abbreviations in **T**, Let:

| | | |
|---|---|---|
| $\mathrm{R}t\mathcal{A}$ | be | $\mathcal{A}_A$ , for $A$ equal to $t$ . |
| $\forall t\mathrm{R}t\mathcal{A}$ | be | $\prod\mathcal{A}$. |

We proof that R1, …, R9 are theorems of **T**.

**Theorem 4.1:** *R1 is a theorem of T.*

Proof:

| | |
|---|---|
| $(\neg\mathcal{A})_t\equiv\neg(\mathcal{A}_t)$ | abbreviation, |
| $\mathrm{R}t\neg\mathcal{A}\equiv\neg\mathrm{R}t\mathcal{A}$ | abbreviation. |

**Theorem 4.2:** *R2 is a theorem of T.*

Proof:

| | |
|---|---|
| $(\mathcal{A}\supset\mathcal{B})_t\supset(\mathcal{A}_t\supset\mathcal{B}_t)$ | abbreviation, |
| $\mathrm{R}t((\mathcal{A}\supset\mathcal{B})\supset(\mathrm{R}t\mathcal{A}\supset\mathrm{R}t\mathcal{B}))$ | abbreviation. |

**Theorem 4.3:** *R3 is a theorem of T.*

Proof:

| | |
|---|---|
| $((\mathcal{A}_t\supset\mathcal{B}_t)\supset(\mathcal{B}_t\supset\mathcal{C}_t))\supset(\mathcal{A}_t\supset\mathcal{C}_t)$ | theorem, |
| $\mathrm{R}t(((\mathcal{A}\supset\mathcal{B})\supset(\mathcal{B}\supset\mathcal{C}))\supset(\mathcal{A}\supset\mathcal{C}))$ | abbreviation. |

**Theorem 4.4:** *R4 is a theorem of T.*

Proof:

| | |
|---|---|
| $(\mathcal{A}_t\supset\neg\mathcal{A}_t)\supset\mathcal{B}_t$ | theorem, |
| $\mathrm{R}t((\mathcal{A}\supset\neg\mathcal{A})\supset\mathcal{B})$ | abbreviation. |

**Theorem 4.5:** *R5 is a theorem of T.*

Proof:

| | |
|---|---|
| $(\neg\mathcal{A}_t\supset\mathcal{A}_t)\supset\mathcal{A}_t$ | theorem, |
| $Rt((\neg\mathcal{A}\supset\mathcal{A})\supset\mathcal{A})$ | abbreviation. |

**Theorem 4.6:** *R6 is a theorem of T.*

Proof:

| | |
|---|---|
| $\mathcal{A}\supset\mathcal{A}$ | theorem, |
| $\prod\mathcal{A}\supset\mathcal{A}$ | abbreviation, |
| $\forall t Rt\mathcal{A}\supset\mathcal{A}$ | abbreviation. |

**Theorem 4.7:** *R7 is a theorem of T.*

Proof:

| | |
|---|---|
| $\forall t\forall n\exists t'(t+n=t')$ | theorem, |
| $\forall t\forall n\exists t'(\mathcal{A}_{t+n}\equiv\mathcal{A}_{t'})$ | theorem, |
| $\forall t\forall n\exists t'(R(t+n)\mathcal{A}\equiv Rt'\mathcal{A})$ | abbreviation, |

$\mathcal{A}$ is a generic sentence, thus is freely replaceable. Let the universal quantification of $\mathcal{A}$ mean that $\mathcal{A}$ is freely replaceable. So, we can write:

| | |
|---|---|
| $\forall t\forall n\exists t'\forall\mathcal{A}(R(t+n)\mathcal{A}\equiv Rt'\mathcal{A})$ | abbreviation. |

**Theorem 4.8:** *R8 is a theorem of T.*

Proof:

| | |
|---|---|
| $\forall t\forall n\exists t'(t'+n=t)$ | theorem, |
| $\forall t\forall n\exists t'(\mathcal{A}_{t'+n}\equiv\mathcal{A}_{t})$ | theorem, |
| $\forall t\forall n\exists t'(R(t'+n)\mathcal{A}\equiv Rt\mathcal{A})$ | abbreviation; |

$\mathcal{A}$ is a generic sentence, thus is freely replaceable. Let the universal quantification of $\mathcal{A}$ mean that $\mathcal{A}$ is freely replaceable. So, we can write:

| | |
|---|---|
| $\forall t\forall n\exists t'\forall\mathcal{A}(R(t'+n)\mathcal{A}\equiv Rt\mathcal{A})$ | abbreviation. |

**Theorem 4.9:** *R9 is a theorem of T.*

Proof:

| | |
|---|---|
| $(\mathcal{B}_t \equiv \mathcal{B}_{t'}) \equiv (\mathcal{B}_t \equiv \mathcal{B}_{t'})$ | theorem, |
| $t = h, t'$ | always possible by definition in **V**, |
| $(\mathcal{B}_{h,t'} \equiv \mathcal{B}_{t'}) \equiv (\mathcal{B}_t \equiv \mathcal{B}_{t'})$ | theorem, |
| $(\mathcal{B}_h \equiv \mathcal{B})_{t'} \equiv (\mathcal{B}_t \equiv \mathcal{B}_{t'})$ | abbreviation. |

The sentence $\mathcal{A}_{t'} \equiv (\mathcal{B}_t \equiv \mathcal{B}_{t'})$ becomes a theorem only when we substitute some sentences, e.g. $\mathcal{B}_h \equiv \mathcal{B}$ for $\mathcal{A}$, but it is not a theorem in general. Instead, for opportune substitutions of $\mathcal{A}$ (for inst. $\mathcal{B}_h \equiv \mathcal{B}$), we can substitute $\mathcal{A}$ freely. Let the universal quantification of a sentence $\mathcal{A}$ mean that $\mathcal{A}$ is replaceable freely. Let the existential quantification of a sentence $\mathcal{A}$ mean that $\mathcal{A}$ is replaceable by some sentences. So, we can write:

$\exists\mathcal{A}(\mathcal{A}_{t'} \equiv (\mathcal{B}_t \equiv \mathcal{B}_{t'}))$ abbreviation.

Observe also that the choise of $\mathcal{A}$ depends from $t$ but not from $t'$ (e. g. $\mathcal{B}_h \equiv \mathcal{B}$ contains the index $h$ which is part of the couple $h, t'$ that corresponds to $t$ but it does not contain $t'$ ), therefore the scope of the existential quantifier of $\mathcal{A}$ has to be contained in the scope of the universal quantifier of $t$ and it has to contain the scope of the universal quantifier of $t'$ . So we can write:

$\forall t \exists \mathcal{A} \forall t' (\mathcal{A}_{t'} \equiv \forall \mathcal{B}(\mathcal{B}_t \equiv \mathcal{B}_{t'}))$ abbreviation.

As R1, …, R9 are theorem of **T**, the Łoś's time logic is a sub-logic of **T** and the Łoś's operator **R** is a time connective of **T**.

## 5. A Prior's Time Logic

Since 1953 A. Prior obtained various results. We consider one of the formal structures studied by him. Call with Mc Taggart *A*-series the time series "past - present - future" and *B*-series the time series "before - after". Prior distinguishes four time involvement degrees whose relations between the *A*-series logic and *B*-series logic are representable.

*5.a First Time Involvement Degree.*

A formal structure to represent the first degree can be obtained by adding to the first order predicative calculus with identity these two axioms:

UR1: $\mathrm{R}t\,\neg p \equiv \neg \mathrm{R}tp$,
UR2: $\mathrm{R}t(p \supset q) \supset (\mathrm{R}tp \supset \mathrm{R}tq)$,

and these four abbreviations; let:

| | | | |
|---|---|---|---|
| $\mathrm{R}t\mathrm{F}p$ | be | $\exists t'(Utt' \wedge \mathrm{R}t'p)$ | (URF), |
| $\mathrm{R}t\mathrm{P}p$ | be | $\exists t'(Ut't \wedge \mathrm{R}t'p)$ | (URP), |
| $\mathrm{G}\,p$ | be | $\neg\mathrm{F}\neg p$, | |
| $\mathrm{H}\,p$ | be | $\neg\mathrm{P}\neg p$, | |

Obviously $p$ and $q$ have to be interpreted as sentential variables of the standard sentence logic and $t$ and $t'$ as time numeric measures. Prior distinguishes the sentences of his time logic in tensed sentences and untensed sentences. The sentential variables are tensed sentences. If $\varphi$ and $\psi$ are tensed sentences, then $\varphi\supset\psi$, $\neg\varphi$, $\mathrm{F}\varphi$ and $\mathrm{P}\varphi$ are tensed sentences. The operators G, H and the connectives $\equiv$, $\vee$ and $\wedge$ are introduced by using the previous abbreviations. $t=t'$ and $Utt'$ are untensed sentences. If $\varphi$ is a tensed sentence, then $\mathrm{R}t\varphi$ is an untensed sentence. If $\alpha$ and $\beta$ are untensed sentences, then $\alpha\supset\beta$, $\neg\alpha$, $\forall t\alpha$ are untensed sentences. It is not possible to have mixed sentences as “$\mathrm{R}tUtt'$”.

To represent the Prior’s time logic in **T**, put the following abbreviations in **T**, Let:

| | | |
|---|---|---|
| $Utt'$ | be | $t< t'$, where $t$ and $t'$ are terms of **V**, |
| $t=t'$ | be | $t= t'$, where $t$ and $t'$ are terms of **V**, |
| $\forall t\alpha$ | be | $\prod\alpha$, when $\alpha$ is a sentence of K. |

We proof that UR1, UR2 are theorems of T.

**Theorem 5.1:** *UR1 is a theorem of T.*

Proof:

For identity with the theorem 4.1.

**Theorem 5.2:** *UR2 is a theorem of T.*

Proof:

For identity with the theorem 4.2.

As UR1, UR2 are theorems of T, the first time involvement degree of the Prior's time logic is a sub-logic of T and the Prior's operator R is a time operator of T. Give the following semantic interpretations. Let:

| | | |
|---|---|---|
| *Utt'* | be | "the time interval *t* is before *t'* ", |
| R*t* P*p* | be | "*p* is true in some time interval before *t* ", |
| R*t* F*p* | be | "*p* is true in some time interval after *t* ", |
| R*t* H*p* | be | "*p* is true in every time interval before *t* ", |
| R*t* G*p* | be | "*p* is true in every time interval after *t* ". |

In T we can put the following interpretation where (0, …) is the known individual constant of V. Let:

| | | |
|---|---|---|
| P*p* | be | "*p* is true in some time interval before 0, … ", |
| F*p* | be | "*p* is true in some time interval before 0, … ", |
| H*p* | be | "*p* is true in every time interval before 0, … ", |
| G*p* | be | "*p* is true in every time interval before 0, … ", |
| R(0,…)*p* | be | "*p* now", |
| NW*p* | be | R(0,…)*p*. |

Obviously:

| | | |
|---|---|---|
| $R(0,\ldots)Pp$ | be | $Pp$, |
| $R(0,\ldots)Fp$ | be | $Fp$, |
| $R(0,\ldots)Hp$ | be | $Hp$, |
| $R(0,\ldots)Gp$ | be | $Gp$, |
| $Pp$ | be | $(\exists z_n)((z_n<0)\wedge(p\equiv Rz_np))$, |
| $Fp$ | be | $(\exists z_n)((z_n>0)\wedge(p\equiv Rz_np))$, |
| $Hp$ | be | $(\forall z_n)((z_n<0)\wedge(p\supset Rz_np))$, |
| $Gp$ | be | $(\forall z_n)((z_n>0)\wedge(p\supset Rz_np))$, |
| $RtPp$ | be | $(\exists z_n)((z_n<t)\wedge(p\equiv Rz_np))$, |
| $RtFp$ | be | $(\exists z_n)((z_n>t)\wedge(p\equiv Rz_np))$, |
| $RtHp$ | be | $(\forall z_n)((z_n<t)\wedge(p\supset Rz_np))$, |
| $RtGp$ | be | $(\forall z_n)((z_n>t)\wedge(p\supset Rz_np))$. |

*5.b Second Time Involvement Degree.*

To represent the second time involvement level of the Prior's logic we add to the structure of the previous sub-section the following three axioms:

UR3: $\forall tRtp\supset p$,
UR4: $\forall tRtp\supset Rt'\forall tRtp$,
UR5: $Rtp\supset Rt'Rtp$,

and the inference outline:

RT: $\mapsto\alpha \rightarrow \mapsto Rt\alpha$.

Put the following abbreviation. Let:

| | | |
|---|---|---|
| $L\, p$ | be | $\Pi\, p$. |

We proof that UR3, UR4, UR5 are theorems of T.

**Theorem 5.3:** *UR3 is a theorem of T.*

Proof:

For identity with the theorem 4.6.

**Theorem 5.4:** *UR4 is a theorem of T.*

Proof:

| | |
|---|---|
| $\prod\mathcal{A}\supset(\prod\mathcal{A})_{t'}$ | axiom outline T1. |

**Theorem 5.5:** *UR5 is a theorem of T.*

Proof:

| | |
|---|---|
| $\mathcal{A}_t\supset(\mathcal{A}_t)_{t'}$ | axiom outline T1. |

We proof that RT is an inference outline true in T:

**Metatheorem 5.1:** *RT is an inference outline true in T.*

Proof:

| | |
|---|---|
| $\mathcal{A}$ | hypothesis, |
| $\mathcal{A}\supset\mathcal{A}_t$ | axiom outline T1, |
| $\mathcal{A}_t$ | for MP, |
| $\mathcal{A}_t$ is deducible from $\mathcal{A}$ | deduction theorem. |

As UR3, UR4, UR5 are theorems of **T** and RT is an inference outline of **T**, the second time involvement degree of the Prior's time logic is a sub-logic of **T**.

*5.c Third Time Involvement Degree.*

To represent the third time involvement level of the Prior's logic we add to the structure of the previous sub-section the axiom:

A6: $\mathrm{R}tt'\equiv t=t'$.

By the previous interpretation of R in the sentence R$t$ $\mathcal{A}$, the semantic interpretation of R$tt'$ is the following:

R$tt'$ means "the time interval $t$ is placed in the time interval $t'$ ",

but this fact is true iff $t$ is identical with $t'$. Hence, we can use A6 as an abbreviation in **T**; therefore also the third time involvement degree of the Prior's time logic is a sub-logic of **T**.

*5.d Fourth Time Involvement Degree.*

To represent the fourth time involvement level of the Prior's logic we add to the structure of the previous sub-section the axiom:

UR6: $Utt' \vee Ut't \vee t = t'$.

**Theorem 5.6:** *UR6 is a theorem of T.*

Proof:

It is a theorem of **V**.

Therefore also the fourth time involvement degree of the Prior's time logic is a sub-logic of T.

*5.e Some considerations*

Remember that L$p$ is $\prod p$. As $t$ is a term and $p$ a sentence, we do not accept formulas as $t \supset p$ differently from Prior. Consequently we refuse all Prior's theorems that contain similar formulas. We can accept as abbreviations Prior's theorems as:

$$Utt' \equiv \mathrm{R}t\mathrm{F}t',$$

but we do not accept theorems as:

$$\exists tt;$$

finally, we put a distinction between time intervals which for us are terms and indexes (obviously, as people can deduce easily from the previous sections, we consider the indexes in T as particular terms) and the sentences. We cannot affirm a term but only a sentence and only sentences can be arguments of the connectives and the scopes of quantifiers.

## 6. Future and Determinism

The greatest objection to the examined Prior's logic is that its theorem:

$$p\supset \mathrm{HF}p,$$

that means "if $p$ is true, then it has been true always which $p$ will be true", is not an affirmation metaphysically neutral because it would imply a deterministic world. To obtain a time logic, which is neutral from a metaphysical standpoint, various solutions have been proposed. In particular, Prior proposed a three-valued logic where the third value is of the indeterminate future sentences.

In **T** we must distinguish the sentence $\mathcal{A}$, which is the abbreviation of $\prod\mathcal{A}$, from the sentence N W$\mathcal{A}$.

In the former case we have:

$$\mathcal{A}\supset \mathrm{HF}\mathcal{A},$$

and, for abbreviation,

$$\prod\mathcal{A}\supset \mathrm{HF}\mathcal{A}$$

that means “if $\mathcal{A}$ is true always (i. e. in every time), then it has been true always that $\mathcal{A}$ will be true” which is true either in a deterministic world or in an indeterministic world.

In the later case we have:

$$NW\mathcal{A}\supset HF\mathcal{A},$$

which means “if $\mathcal{A}$ is true now (i. e. in this instant), then it has been true always that $\mathcal{A}$ will be true”. In effect this sentence is true only in deterministic worlds, but it is not a theorem of T. Anybody could consider the sentence:

$$NW\mathcal{A}\supset HFNW\mathcal{A},$$

which is a theorem of T, but, by considering its abbreviation:

$$\prod NW\mathcal{A}\supset HFNW\mathcal{A},$$

we understend quickly the neutral nature of this affirmation for the determinism question.

We can conclude that T is neutral for the determinism question.

## 7. Conclusion

This is only a first study on some questions in time logic for building a bivalent time logic. Surely, it is incomplete and some parts will be changed in future for various imperfections or, perhaps, mistakes. However, we hope to have demonstrated that the time logic for the indeterministic world can be also 2-valued.

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