Metalogicon (1990) III, 2

Modal Sentence Logic Formulas as Abbreviations of Standard Sentence Logic Formulas

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To the prof. R. Magari for his studies on unification of Mathematics.

1. Introduction

In this work we proof that every modal sentence logic formula, in which there are only standard sentence logic connectives, sentential variables, and the modal ‘connectives’ □ (read “it is necessary that …”) and/or ◇ (read “it is possible that …”), is equivalent to an opportune standard sentence logic formula in which there are only standard sentence logic connectives and sentential variables.

2. Axiomatic Formal Theories

See GRAPPONE [1990] 4 - 6

3. Standard Sentence Logic (*SSL*) as Formal Theory

See GRAPPONE [1990] 6 - 7

4. *SSL* as Interpreted Boole's Algebra

See GRAPPONE [1990] 7 - 11

5. Carnap's Equivalence: Consequences

Carnap CARNAP [1970] affirms that any sentence $\mathcal{A}$ is necessary iff it is a tautology. This affirmation permits us to build the following table:

$\mathcal{A}$	it is necessary that $\mathcal{A}$
tautology	true
neuter formula (*)	false
contradiction	false

Generally the expression "it is necessary that …" is denoted by the symbol " □ " in standard formalization of the modal logic. Therefore we can write:

$\mathcal{A}$	$\Box\mathcal{A}$
tautology	true
neuter formula	false
contradiction	false

If we consider " □ " as a monadic 'connective' applied to $\mathcal{A}$, then this 'connective' trasforms a truth function in a single truth value. In other terms the truth function of $\Box\mathcal{A}$ is done by a single truth value which can be *1* or *0* . But in the former case $\Box\mathcal{A}$ is a tautology by definition and in the later case $\Box\mathcal{A}$ is a contradiction by definition. Therefore, we can write:

(*) MALATESTA, 1988

$\mathcal{A}$	$\Box\mathcal{A}$
tautology	tautology
neuter formula	contradiction
contradiction	contradiction

In this way we can understand the nature of the modal 'connective' □. Really, it is 'meta-connective' because it does not operate on the single truth values of each truth functions as the *SSL* connectives, but on the whole truth functions.

If we want build an eventual 'modal tautology calculus', then, to verify any wff, we must consider not only all the possible combinations of truth values which correspond to the distinct atomic sentences as in *SSL*, but also all the possible combinations of being tautology, neuter formula or contradiction which correspond to the whole truth functions of the distinct atomic sentences.

Example 5.1

So, for example, if in *SSL* we consider the only atomic sentence $\mathcal{A}_1$, then we consider in its truth function only the case in which $\mathcal{A}_1$ is true and the case in which $\mathcal{A}_1$ is false; so, the truth function of $\mathcal{A}_1$ is $\begin{matrix}1\\0\end{matrix}$.

But, if we must consider all the possible combinations of being tautology, neuter formula or contradiction which correspond to the whole truth function of $\mathcal{A}_1$, then we must consider the case in which $\mathcal{A}_1$ is a tautology, the case in which $\mathcal{A}_1$ is a true neuter formula, the case in which $\mathcal{A}_1$ is a false neuter formula, and, finally, the case in which $\mathcal{A}_1$ is a contradiction; so, we have a 'function of truth functions' for $\mathcal{A}_1$, i. e. we have $\begin{matrix}1\\ \\ \begin{matrix}1\\0\end{matrix}\\ \\ 0\end{matrix}$, a column of three truth functions. Observe that every truth function of our column has the least lenght which needes to represent it without ambiguity, so, the tautology and the contradiction are represented by a single truth value (because in these cases all the truth values are equal among them), the neuter formula by two truth values (because the truth function *10101010……* [which is written horizontally] is univocally representable by the truth function *10* when there is only a distinct atomic sentence).

In general call *truth metafunction* a function of truth functions and represent it as a column of truth function, i. e. as a column of columns of truth values. Either the *SSL* connectives or the metaconnective □ can be operate on a truth metafunction.

The *SSL* connectives operate on the single truth value in every truth function in the metafunction. Instead the metaconnective □ operates on each whole function in the metafunction. So, for example we have:

Example 5.2

Consider the formula

$$\neg\Box\neg\mathcal{A}_1.$$

We want verify it. Assign to $\mathcal{A}_1$ its truth metafunction:

	$\neg\Box\neg\mathcal{A}_1$
Tautology	*1*
Neuter formula	*1* *0*
Contradiction	*0*

Calculate the most internal *SSL* connective ¬ in the usual way (for single truth value):

¬	□	¬	$\mathcal{A}_1$
		0	*1*
		0 *1*	*1* *0*
		1	*0*

Calculate the metaconnective □ for single truth function by replacing tautology with tautology, neuter formula with contradiction and contradiction with contradiction as in the previous table:

$\mathcal{A}$	$\Box\mathcal{A}$
tautology neuter formula contradiction	tautology contradiction contradiction

We obtain:

$\neg$	$\Box$	$\neg$	$\mathcal{A}_1$
	0	*0*	*1*
	0	*0*	*1*
	0	*1*	*0*
	1	*1*	*0*

Calculate the most external *SSL* connective ¬ in the usual way (for single truth value):

$\neg$	$\Box$	$\neg$	$\mathcal{A}_1$
1	*0*	*0*	*1*
1	*0*	*0*	*1*
1	*0*	*1*	*0*
0	*1*	*1*	*0*

For a much common convention in modal logics we abbreviate $\neg\Box\neg\mathcal{A}$ with $\Diamond\mathcal{A}$ by using a new metaconnective $\Diamond$. For definition this connective means "it is possible that" as in the great part of the modal logics. Consequently, we can define its truth table in this way:

$\mathcal{A}$	$\Diamond\mathcal{A}$
tautology neuter formula contradiction	tautology tautology contradiction

We give an example of a possible truth metafunction calculus which we build by the previous considerations:

Example 5.3

□	𝒜	∧	◇	𝒜	∧	□	(𝒞	⊃	◇	ℬ)	∨	◇	𝒞
1	*1*	*1*	*1*	*1*	*1*	*1*	*1*	*1*	*1*	*1*	***1***	*1*	*1*
1	*1*	*1*	*1*	*1*	*1*	*1*	*1*	*1*	*1*	*1*	***1***	*1*	*1*
1	*1*	*1*	*1*	*1*	*1*	*1*	*0*	*1*	*1*	*1*	***1***	*1*	*0*
1	*1*	*1*	*1*	*1*	*1*	*1*	*0*	*1*	*1*	*1*	***1***	*0*	*0*
1	*1*	*1*	*1*	*1*	*1*	*1*	*1*	*1*	*1*	*1*	***1***	*1*	*1*
1	*1*	*1*	*1*	*1*	*1*	*1*	*1*	*1*	*1*	*0*	***1***	*1*	*1*
1	*1*	*1*	*1*	*1*	*1*	*1*	*1*	*1*	*1*	*1*	***1***	*1*	*1*
1	*1*	*1*	*1*	*1*	*1*	*1*	*0*	*1*	*1*	*1*	***1***	*1*	*0*
1	*1*	*1*	*1*	*1*	*1*	*1*	*1*	*1*	*1*	*0*	***1***	*1*	*1*
1	*1*	*1*	*1*	*1*	*1*	*1*	*0*	*1*	*1*	*0*	***1***	*1*	*0*
1	*1*	*1*	*1*	*1*	*1*	*1*	*0*	*1*	*1*	*1*	***1***	*0*	*0*
1	*1*	*1*	*1*	*1*	*1*	*1*	*0*	*1*	*1*	*0*	***1***	*0*	*0*
1	*1*	*1*	*1*	*1*	*0*	*0*	*1*	*0*	*0*	*0*	***1***	*1*	*1*
1	*1*	*1*	*1*	*1*	*0*	*0*	*1*	*0*	*0*	*0*	***1***	*1*	*1*
1	*1*	*1*	*1*	*1*	*0*	*0*	*0*	*1*	*0*	*0*	***1***	*1*	*0*
1	*1*	*1*	*1*	*1*	*1*	*1*	*0*	*1*	*0*	*0*	***1***	*0*	*0*
0	*1*	*0*	*1*	*1*	*0*	*1*	*1*	*1*	*1*	*1*	***1***	*1*	*1*
0	*0*	*0*	*1*	*0*	*0*	*1*	*1*	*1*	*1*	*1*	***1***	*1*	*1*
0	*1*	*0*	*1*	*1*	*0*	*1*	*1*	*1*	*1*	*1*	***1***	*1*	*1*
0	*1*	*0*	*1*	*1*	*0*	*1*	*0*	*1*	*1*	*1*	***1***	*1*	*0*
0	*0*	*0*	*1*	*0*	*0*	*1*	*1*	*1*	*1*	*1*	***1***	*1*	*1*
0	*0*	*0*	*1*	*0*	*0*	*1*	*0*	*1*	*1*	*1*	***1***	*1*	*0*
0	*1*	*0*	*1*	*1*	*0*	*1*	*0*	*1*	*1*	*1*	***0***	*0*	*0*
0	*0*	*0*	*1*	*0*	*0*	*1*	*0*	*1*	*1*	*1*	***0***	*0*	*0*
0	*1*	*0*	*1*	*1*	*0*	*1*	*1*	*1*	*1*	*1*	***1***	*1*	*1*
0	*1*	*0*	*1*	*1*	*0*	*1*	*1*	*1*	*1*	*0*	***1***	*1*	*1*
0	*0*	*0*	*1*	*0*	*0*	*1*	*1*	*1*	*1*	*1*	***1***	*1*	*1*
0	*0*	*0*	*1*	*0*	*0*	*1*	*1*	*1*	*1*	*0*	***1***	*1*	*1*
0	*1*	*0*	*1*	*1*	*0*	*1*	*1*	*1*	*1*	*1*	***1***	*1*	*1*
0	*1*	*0*	*1*	*1*	*0*	*1*	*0*	*1*	*1*	*1*	***1***	*1*	*0*
0	*1*	*0*	*1*	*1*	*0*	*1*	*1*	*1*	*1*	*0*	***1***	*1*	*1*
0	*1*	*0*	*1*	*1*	*0*	*1*	*0*	*1*	*1*	*0*	***1***	*1*	*0*
0	*0*	*0*	*1*	*0*	*0*	*1*	*1*	*1*	*1*	*1*	***1***	*1*	*1*
0	*0*	*0*	*1*	*0*	*0*	*1*	*0*	*1*	*1*	*1*	***1***	*1*	*0*
0	*0*	*0*	*1*	*0*	*0*	*1*	*1*	*1*	*1*	*0*	***1***	*1*	*1*
0	*0*	*0*	*1*	*0*	*0*	*1*	*0*	*1*	*1*	*0*	***1***	*1*	*0*
0	*1*	*0*	*1*	*1*	*0*	*1*	*0*	*1*	*1*	*1*	***0***	*0*	*0*
0	*1*	*0*	*1*	*1*	*0*	*1*	*0*	*1*	*1*	*0*	***0***	*0*	*0*
0	*0*	*0*	*1*	*0*	*0*	*1*	*0*	*1*	*1*	*1*	***0***	*0*	*0*
0	*0*	*0*	*1*	*0*	*0*	*1*	*0*	*1*	*1*	*0*	***0***	*0*	*0*
0	*1*	*0*	*1*	*1*	*0*	*0*	*1*	*0*	*0*	*0*	***1***	*1*	*1*
0	*0*	*0*	*1*	*0*	*0*	*0*	*1*	*0*	*0*	*0*	***1***	*1*	*1*
0	*1*	*0*	*1*	*1*	*0*	*0*	*1*	*0*	*0*	*0*	***1***	*1*	*1*
0	*1*	*0*	*1*	*1*	*0*	*0*	*0*	*1*	*0*	*0*	***1***	*1*	*0*
0	*0*	*0*	*1*	*0*	*0*	*0*	*1*	*0*	*0*	*0*	***1***	*1*	*1*
0	*0*	*0*	*1*	*0*	*0*	*0*	*0*	*1*	*0*	*0*	***1***	*1*	*0*
0	*1*	*0*	*1*	*1*	*0*	*1*	*0*	*1*	*0*	*0*	***0***	*0*	*0*
0	*0*	*0*	*1*	*0*	*0*	*1*	*0*	*1*	*0*	*0*	***0***	*0*	*0*
0	*0*	*0*	*0*	*0*	*0*	*1*	*1*	*1*	*1*	*1*	***1***	*1*	*1*
0	*0*	*0*	*0*	*0*	*0*	*1*	*1*	*1*	*1*	*1*	***1***	*1*	*1*
0	*0*	*0*	*0*	*0*	*0*	*1*	*0*	*1*	*1*	*1*	***1***	*1*	*0*
0	*0*	*0*	*0*	*0*	*0*	*1*	*1*	*1*	*1*	*1*	***0***	*0*	*0*
0	*0*	*0*	*0*	*0*	*0*	*1*	*1*	*1*	*1*	*1*	***1***	*1*	*1*
0	*0*	*0*	*0*	*0*	*0*	*1*	*1*	*1*	*1*	*0*	***1***	*1*	*1*
0	*0*	*0*	*0*	*0*	*0*	*1*	*1*	*1*	*1*	*1*	***1***	*1*	*1*
0	*0*	*0*	*0*	*0*	*0*	*1*	*0*	*1*	*1*	*1*	***1***	*1*	*0*
0	*0*	*0*	*0*	*0*	*0*	*1*	*1*	*1*	*1*	*0*	***1***	*1*	*1*
0	*0*	*0*	*0*	*0*	*0*	*1*	*0*	*1*	*1*	*0*	***1***	*1*	*0*
0	*0*	*0*	*0*	*0*	*0*	*1*	*0*	*1*	*1*	*1*	***0***	*0*	*0*
0	*0*	*0*	*0*	*0*	*0*	*1*	*0*	*1*	*1*	*0*	***0***	*0*	*0*
0	*0*	*0*	*0*	*0*	*0*	*0*	*1*	*0*	*0*	*0*	***1***	*1*	*1*
0	*0*	*0*	*0*	*0*	*0*	*0*	*1*	*0*	*0*	*0*	***1***	*1*	*1*
0	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*1*	*0*	*0*	***1***	*1*	*0*
0	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*1*	*0*	*0*	***0***	*0*	*0*

This truth metafunction calculus will be described in the Proceedings of the Congress "Logica 91" organized by the Czechoslovak Academy of Science. GRAPPONE [1991] (preprint)

6. Reduction of the Modal Calculate to the Standard Tautology Calculus

Let $\mathcal{B}$ be a formula in which there are *SSL* atomic sentences, *SSL* connectives and the modal operators $\Box$ and $\Diamond$. On $\mathcal{B}$ we apply the following algorithm $\aleph(\mathcal{B})$ where $\mathcal{A}_n$ is an atomic formula at will, and $\mathcal{A}$ and C are generic subformulas of $\mathcal{B}$ without modal operators.

STEP 1: Replace every $\mathcal{A}_n$ with $\mathcal{A}_{2n}$.

STEP 2: Replace every $\Box\neg\mathcal{A}$ with $\neg\Diamond\mathcal{A}$.

STEP 3: Replace every $\Diamond\neg\mathcal{A}$ with $\neg\Box\mathcal{A}$.

STEP 4: Replace every $\Box\mathcal{A}_{2n}$ with $\left(\mathcal{A}_{2n-1} \wedge \mathcal{A}_{2n}\right)$.

STEP 5: Replace every $\Diamond\mathcal{A}_{2n}$ with $\left(\mathcal{A}_{2n-1} \vee \mathcal{A}_{2n}\right)$.

STEP 6: Replace every $\Box\left(\mathcal{A}_{2n-1} \wedge \mathcal{A}_{2n}\right)$ with $\left(\mathcal{A}_{2n-1} \wedge \mathcal{A}_{2n}\right)$.

STEP 7: Replace every $\Diamond\left(\mathcal{A}_{2n-1} \wedge \mathcal{A}_{2n}\right)$ with $\left(\mathcal{A}_{2n-1} \wedge \mathcal{A}_{2n}\right)$.

STEP 8: Replace every $\Box\left(\mathcal{A}_{2n-1} \vee \mathcal{A}_{2n}\right)$ with $\left(\mathcal{A}_{2n-1} \vee \mathcal{A}_{2n}\right)$.

STEP 9: Replace every $\Diamond\left(\mathcal{A}_{2n-1} \vee \mathcal{A}_{2n}\right)$ with $\left(\mathcal{A}_{2n-1} \vee \mathcal{A}_{2n}\right)$.

STEP 10: If any step among 2, …, 9 has been executed, then go to STEP 2.

STEP 11: Replace every $\Box(\mathcal{A}\wedge C)$ with $\Box\mathcal{A}\wedge\Box C$.

STEP 12: Replace every $\Diamond(\mathcal{A}\wedge C)$ with $\Diamond\mathcal{A}\wedge\Diamond C$ when $\mathcal{A}\wedge C$ is not contradictory, otherwise with $\mathcal{A}\wedge C$.

STEP 13: Replace every $\Box(\mathcal{A}\vee C)$ with $\Box\mathcal{A}\vee\Box C$ when $\mathcal{A}\vee C$ is not tautology, otherwise with $\mathcal{A}\vee C$.

STEP 14: Replace every $\Diamond(\mathcal{A}\vee\mathcal{C})$ with $\Diamond\mathcal{A}\vee\Diamond\mathcal{C}$.

STEP 15: Replace every $\Box(\mathcal{A}\supset\mathcal{C})$ with $\Diamond\mathcal{A}\supset\Box\mathcal{C}$ when $\mathcal{A}\supset\mathcal{C}$ is not tautology, otherwise with $\mathcal{A}\supset\mathcal{C}$.

STEP 16: Replace every $\Diamond(\mathcal{A}\supset\mathcal{C})$ with $\Box\mathcal{A}\supset\Diamond\mathcal{C}$.

STEP 17: Replace every $\Box(\mathcal{A}\equiv\mathcal{C})$ with $(\Diamond\mathcal{A}\supset\Box\mathcal{C})\wedge(\Diamond\mathcal{C}\supset\Box\mathcal{A})$ when $\mathcal{A}\equiv\mathcal{C}$ is not tautology, otherwise with $\mathcal{A}\equiv\mathcal{C}$.

STEP 18: Replace every $\Diamond(\mathcal{A}\equiv\mathcal{C})$ with $(\Box\mathcal{A}\supset\Diamond\mathcal{C})\wedge(\ \Box\mathcal{C}\supset\Diamond\mathcal{A})$ when $\mathcal{A}\equiv\mathcal{C}$ is not contradictory, otherwise with $\mathcal{A}\equiv\mathcal{C}$.

STEP 19: If any step among 11, …, 18 has been executed, go to STEP 2.

The algorithm $\aleph$ transform every sentence $\mathcal{B}$ with modal operators in a sentence $\aleph(\mathcal{B})$ without modal operators. We proof the following metatheorem:

Metatheorem 6.1: $\aleph(\mathcal{B})$ *is equivalent to* $\mathcal{B}$.

Proof:

The proof of the theorem is equivalent to proof that every substitution in $\aleph$ corresponds to an equivalence.

STEPS 1,4,5: The sets $\langle\Diamond\mathcal{A}_n,\ \mathcal{A}_n,\ \Box\mathcal{A}_n\rangle$ and $\langle\mathcal{A}_{2n-1}\vee\mathcal{A}_{2n},\ \mathcal{A}_{2n},\ \mathcal{A}_{2n-1}\wedge\mathcal{A}_{2n}\rangle$ have the same logical structure because in both the third element implies the second element and the second element implies the first element.

STEPS 2, 3: The necessity of not $\mathcal{A}$ is the impossibility of $\mathcal{A}$ and the possibility of not $\mathcal{A}$ is the non-necessity of $\mathcal{A}$.

STEPS 6,7,8,9: The results of the modal operators are only tautologies or contradictions. But the modal operators $\Box$ and $\Diamond$ are equivalent to identity operator $\mathbb{I}$ when its arguments are only tautologies or contradictions. In fact we have:

$\mathcal{A}$	$\mathbb{I}\,\mathcal{A}$
tautology neuter formula contradiction	tautology neuter formula contradiction

$\mathcal{A}$	$\Box\mathcal{A}$
tautology neuter formula contradiction	tautology contradiction contradiction

$\mathcal{A}$	$\Diamond\mathcal{A}$
tautology neuter formula contradiction	tautology tautology contradiction

but, if we consider only tautologies and contradictions as arguments, then we have:

$\mathcal{A}$	$\mathbb{I}\,\mathcal{A}$
tautology contradiction	tautology contradiction

$\mathcal{A}$	$\Box\mathcal{A}$
tautology contradiction	tautology contradiction

$\mathcal{A}$	$\Diamond\mathcal{A}$
tautology contradiction	tautology contradiction

STEP 11: A necessary logical product is equivalent to the logical product of necessary terms.

STEP 12: A possible logical product is equivalent to the logical product of possible terms when it is not a contradiction, otherwise it is equivalent to the same logical product.

STEPS 13, 14: From the steps 11, 12 by considering that $\mathcal{A}\vee\mathcal{B}$ is equivalent to $\neg(\neg\mathcal{A}\vee\neg\mathcal{B})$.

STEPS 15, 16: From the steps 13, 14 by considering that $\mathcal{A}\supset\mathcal{B}$ is equivalent to $\neg\mathcal{A}\vee\mathcal{B}$.

STEPS 17, 18: From the steps 11, 12, 15, 16, by considering that $\mathcal{A}\equiv\mathcal{B}$ is equivalent to $(\mathcal{A}\supset\mathcal{B})\wedge(\mathcal{B}\supset\mathcal{A})$. *c.d.d.*

We give an example of application of $\aleph$:

Example 6.1

$\Box\mathcal{A}\wedge\Diamond\mathcal{A}\wedge\Box(\mathcal{C}\supset\Diamond\mathcal{B})\vee\Diamond\mathcal{C}$

$\Box\mathcal{A}_1\wedge\Diamond\mathcal{A}_1\wedge\Box(\mathcal{A}_3\supset\Diamond\mathcal{A}_2)\vee\Diamond\mathcal{A}_3$

$\Box\mathcal{A}_2\wedge\Diamond\mathcal{A}_2\wedge\Box(\mathcal{A}_6\supset\Diamond\mathcal{A}_4)\vee\Diamond\mathcal{A}_6$ STEP 1

$(\mathcal{A}_1\wedge\mathcal{A}_2)\wedge\Diamond\mathcal{A}_2\wedge\Box(\mathcal{A}_6\supset\Diamond\mathcal{A}_4)\vee\Diamond\mathcal{A}_6$ STEP 4

$(\mathcal{A}_1\wedge\mathcal{A}_2)\wedge(\mathcal{A}_1\vee\mathcal{A}_2)\wedge\Box(\mathcal{A}_6\supset(\mathcal{A}_3\vee\mathcal{A}_4))\vee(\mathcal{A}_5\vee\mathcal{A}_6)$ STEP 5

$(\mathcal{A}_1\wedge\mathcal{A}_2)\wedge(\mathcal{A}_1\vee\mathcal{A}_2)\wedge(\Diamond\mathcal{A}_6\supset\Box(\mathcal{A}_3\vee\mathcal{A}_4))\vee(\mathcal{A}_5\vee\mathcal{A}_6)$ STEP 15

$(\mathcal{A}_1\wedge\mathcal{A}_2)\wedge(\mathcal{A}_1\vee\mathcal{A}_2)\wedge(\Diamond\mathcal{A}_6\supset(\mathcal{A}_3\vee\mathcal{A}_4))\vee(\mathcal{A}_5\vee\mathcal{A}_6)$ STEP 8

$(\mathcal{A}_1\wedge\mathcal{A}_2)\wedge(\mathcal{A}_1\vee\mathcal{A}_2)\wedge((\mathcal{A}_5\vee\mathcal{A}_6)\supset(\mathcal{A}_3\vee\mathcal{A}_4))\vee(\mathcal{A}_5\vee\mathcal{A}_6)$ STEP 5

So, an opportune *SSL* wff corresponds to every formula of the standard modal sentences logics by $\aleph$, i. e. we can affirm that, in sentence logic, every sentence with modal operators is the abbreviation of a sentence without modal operators. But there is the problem that there are more distinct formal theories to represent the modal logic, instead all the formal theories that represent *SSL* are equivalent among them. In the next paragraph we shall see that all the theorems of the most important formal theories that represent the sentence modal logic are transformed in tautologies by $\aleph$.

7. Modal logics as particular cases of *SSL*.

Consider the most common axiom systems which represent the modal logic MATERNA [1990] (preprint):

System M:

A1: $\Box\mathcal{A}\supset\mathcal{A}$;

A3: $\Box(\mathcal{A}\supset\mathcal{B})\supset(\Box\mathcal{A}\supset\Box\mathcal{B})$;

System S4:

A1: $\Box\mathcal{A}\supset\mathcal{A}$;

A3: $\Box(\mathcal{A}\supset\mathcal{B})\supset(\Box\mathcal{A}\supset\Box\mathcal{B})$;

A4: $\Box\mathcal{A}\supset\Box\Box\mathcal{A}$;

System 'Brouwer':

A1: $\Box\mathcal{A}\supset\mathcal{A}$;

A3: $\Box(\mathcal{A}\supset\mathcal{B})\supset(\Box\mathcal{A}\supset\Box\mathcal{B})$;

$\mathcal{A}\supset\Box\Diamond\mathcal{A}$;

System S5:

A1: $\Box\mathcal{A}\supset\mathcal{A}$;

A2: $\neg\Box\mathcal{A}\supset\Box\neg\Box\mathcal{A}$;

A3: $\Box(\mathcal{A}\supset\mathcal{B})\supset(\Box\mathcal{A}\supset\Box\mathcal{B})$.

In the above systems the inference rules are MP and:

IM: $\mathcal{A} \mapsto \Box\mathcal{A}$.

In the above system there are also the abbreviation:

$\Diamond\mathcal{A}$ curtails $\neg\Box\neg\mathcal{A}$.

We proof that every theorem which is deduced from M, S4, Brouwer, S5 by MP and IM is a tautology in *SSL*, in particular, they are theorems of the formal theory **P**.

A1: $\Box\mathcal{A}\supset\mathcal{A}$

$\Box\mathcal{A}_2\supset\mathcal{A}_2$ STEP 1

$(\mathcal{A}_1\wedge\mathcal{A}_2)\supset\mathcal{A}_2$ STEP 4

1	*1*	*1*	***1***	*1*
1	*0*	*0*	***1***	*0*
0	*0*	*1*	***1***	*1*
0	*0*	*0*	***1***	*0*

A2: $\neg\Box\mathcal{A}\supset\Box\neg\Box\mathcal{A}$

$\neg\Box\mathcal{A}_2\supset\Box\neg\Box\mathcal{A}_2$ STEP 1

$\neg(\mathcal{A}_1\wedge\mathcal{A}_2)\supset\Box\neg(\mathcal{A}_1\wedge\mathcal{A}_2)$ STEP 4

$\neg(\mathcal{A}_1\wedge\mathcal{A}_2)\supset\neg\Diamond(\mathcal{A}_1\wedge\mathcal{A}_2)$ STEP 2

$\neg(\mathcal{A}_1\wedge\mathcal{A}_2)\supset\neg(\mathcal{A}_1\wedge\mathcal{A}_2)$ STEP 7

0	*1*	*1*	*1*	***1***	*0*	*1*	*1*	*1*
1	*0*	*0*	*1*	***1***	*1*	*0*	*0*	*1*
1	*1*	*0*	*0*	***1***	*1*	*1*	*0*	*0*
1	*0*	*0*	*0*	***1***	*1*	*0*	*0*	*0*

A3: $\Box(\mathcal{A}\supset\mathcal{B})\supset(\Box\mathcal{A}\supset\Box\mathcal{B})$

$\Box(\mathcal{A}_2\supset\mathcal{A}_4)\supset(\Box\mathcal{A}_2\supset\Box\mathcal{A}_4)$ STEP 1

$\Box(\mathcal{A}_2\supset\mathcal{A}_4)\supset((\mathcal{A}_1\wedge\mathcal{A}_2)\supset(\mathcal{A}_3\wedge\mathcal{A}_4))$ STEP 4

$(\Diamond\mathcal{A}_2\supset\Box\mathcal{A}_4)\supset((\mathcal{A}_1\wedge\mathcal{A}_2)\supset(\mathcal{A}_3\wedge\mathcal{A}_4))$ STEP 15

$(\Diamond\mathcal{A}_2\supset(\mathcal{A}_3\wedge\mathcal{A}_4))\supset((\mathcal{A}_1\wedge\mathcal{A}_2)\supset(\mathcal{A}_3\wedge\mathcal{A}_4))$ STEP 4

$((\mathcal{A}_1\vee\mathcal{A}_2)\supset(\mathcal{A}_3\wedge\mathcal{A}_4))\supset((\mathcal{A}_1\wedge\mathcal{A}_2)\supset(\mathcal{A}_3\wedge\mathcal{A}_4))$ STEP 5

1	*1*	*1*	*1*	*1*	*1*	*1*	***1***	*1*	*1*	*1*	*1*	*1*	*1*	*1*
0	*1*	*1*	*1*	*1*	*1*	*1*	***1***	*0*	*0*	*1*	*1*	*1*	*1*	*1*
1	*1*	*0*	*1*	*1*	*1*	*1*	***1***	*1*	*0*	*0*	*1*	*1*	*1*	*1*
0	*0*	*0*	*1*	*1*	*1*	*1*	***1***	*0*	*0*	*0*	*1*	*1*	*1*	*1*
1	*1*	*1*	*0*	*0*	*0*	*1*	***1***	*1*	*1*	*1*	*0*	*0*	*0*	*1*
0	*1*	*1*	*0*	*0*	*0*	*1*	***1***	*0*	*0*	*1*	*1*	*0*	*0*	*1*
1	*1*	*0*	*0*	*0*	*0*	*1*	***1***	*1*	*0*	*0*	*1*	*0*	*0*	*1*
0	*0*	*0*	*1*	*0*	*0*	*1*	***1***	*0*	*0*	*0*	*1*	*0*	*0*	*1*
1	*1*	*1*	*0*	*1*	*0*	*0*	***1***	*1*	*1*	*1*	*0*	*1*	*0*	*0*
0	*1*	*1*	*0*	*1*	*0*	*0*	***1***	*0*	*0*	*1*	*1*	*1*	*0*	*0*
1	*0*	*0*	*1*	*0*	*0*	*1*	***1***	*1*	*0*	*0*	*1*	*1*	*0*	*0*
0	*0*	*0*	*1*	*1*	*0*	*0*	***1***	*0*	*0*	*0*	*1*	*1*	*0*	*0*
1	*1*	*1*	*0*	*0*	*0*	*0*	***1***	*1*	*1*	*1*	*0*	*0*	*0*	*0*
0	*1*	*1*	*0*	*0*	*0*	*0*	***1***	*0*	*0*	*1*	*1*	*0*	*0*	*0*
1	*1*	*0*	*0*	*0*	*0*	*0*	***1***	*1*	*0*	*0*	*1*	*0*	*0*	*0*
0	*0*	*0*	*1*	*0*	*0*	*0*	***1***	*0*	*0*	*0*	*1*	*0*	*0*	*0*

A4: $\Box\mathcal{A}\supset\Box\Box\mathcal{A}$

$\Box\mathcal{A}_2\supset\Box\Box\mathcal{A}_2$ STEP 1

$(\mathcal{A}_1\wedge\mathcal{A}_2)\supset\Box(\mathcal{A}_1\wedge\mathcal{A}_2)$ STEP 4

$(\mathcal{A}_1\wedge\mathcal{A}_2)\supset(\mathcal{A}_1\wedge\mathcal{A}_2)$ STEP 6

1	*1*	*1*	***1***	*1*	*1*	*1*
0	*0*	*1*	***1***	*0*	*0*	*1*
1	*0*	*0*	***1***	*1*	*0*	*0*
0	*0*	*0*	***1***	*0*	*0*	*0*

$\mathcal{A}\supset\Box\Diamond\mathcal{A}$

$\mathcal{A}_2\supset\Box\Diamond\mathcal{A}_2$ STEP 1

$\mathcal{A}_2\supset\Box(\mathcal{A}_1\vee\mathcal{A}_2)$ STEP 5

$\mathcal{A}_2\supset(\mathcal{A}_1\vee\mathcal{A}_2)$ STEP 8

1	***1***	*1*	*1*	*1*
1	***1***	*0*	*1*	*1*
0	***1***	*1*	*1*	*0*
0	***1***	*0*	*0*	*0*

We can see easily that in **P** (the formal theory which we use to represent the standard sentence logic, see the paragraph 3) the Herbrand's deduction meta-theorem [Herbrand, 1930] is true. We write it in this form:

$$\text{If } \mathcal{A}\mapsto\mathcal{B}\text{, then } \mapsto(\mathcal{A}\supset\mathcal{B}).$$

In $\mathbf{P}(\mathrm{u},\mathrm{n})$, i. e. the formal theory **P** with the two modal operators, we must write the Herbrand's deduction theorems in this strong form:

$$\mathcal{A}\mapsto\mathcal{B} \text{ iff } \mapsto(\Box\mathcal{A}\supset\mathcal{B}).$$

because, given $\mathcal{A}\mapsto\mathcal{B}$, $\mathcal{B}$ can be deduced from $\mathcal{A}$ iff $\mathcal{A}$ is necessary and it implies $\mathcal{B}$, and we have supposed that $\Box$ means "it is necessary that … " in $\mathbf{P}(\mathrm{u},\mathrm{n})$. So, we can show the following proof:

IM: $\mathcal{A}\mapsto\Box\mathcal{A}$

$\Box\mathcal{A}\supset\Box\mathcal{A}$ Herbrand's deduction meta-theorem

$\Box\mathcal{A}_2\supset\Box\mathcal{A}_2$ STEP 1

$(\mathcal{A}_1\wedge\mathcal{A}_2)\supset(\mathcal{A}_1\wedge\mathcal{A}_2)$ STEP 4

1	*1*	*1*	***1***	*1*	*1*	*1*
0	*0*	*1*	***1***	*0*	*0*	*1*
1	*0*	*0*	***1***	*1*	*0*	*0*
0	*0*	*0*	***1***	*0*	*0*	*0*

As the axioms A1, A2, A3, A4, ## and the inference IM are true in $\mathbf{P}$, i. e. they correspond to tautologies of $\mathbf{P}$, then all the theorems which can be deduced by the modal axiom systems M, S4, Brouwer, S5 are abbreviations of theorems (hence, of tautologies) of $\mathbf{P}$. Also the abbreviation:

$$\Diamond\mathcal{A} \text{ curtails } \neg\Box\neg\mathcal{A}$$

corresponds to a tautology of $\mathbf{P}$. In fact we have:

$\Diamond\mathcal{A}\equiv\neg\Box\neg\mathcal{A}$

$\Diamond\mathcal{A}_2\equiv\neg\Box\neg\mathcal{A}_2$ STEP 1

$\Diamond\mathcal{A}_2\equiv\neg\neg\Diamond\mathcal{A}_2$ STEP 2

$(\mathcal{A}_1\vee\mathcal{A}_2)\equiv\neg\neg(\mathcal{A}_1\vee\mathcal{A}_2)$ STEP 5

1	*1*	*1*	***1***	*1*	*0*	*1*	*1*	*1*
1	*1*	*0*	***1***	*1*	*0*	*1*	*1*	*0*
0	*1*	*1*	***1***	*1*	*0*	*0*	*1*	*1*
0	*0*	*0*	***1***	*0*	*1*	*0*	*0*	*0*

Conclusion

From the previous results we can affirm that does not exist - or should not exist! - a sentence modal logic which is distinct from the standard sentence logic. The modal logic in its natural meaning is based on the truth values 'true' and 'false' and it is not necessary to put a third truth value as it has been done in previous approaches. Others studies are necessary to find if also other 'non-standard logics' are only particular cases of the bivalent standard sentence logic.

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