Metalogicon (1990) III, 1

Bi-logic Nature of the Numeric Calculus

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To the prof. C. D'Amelio for his studies on relation between logical calculus and arithmetical calculus.

1. Introduction

In this paper we study some results that we can obtain by applying the information entropy to study the standard sentence logic (*SSL*). The most important of them is that any interpretation of the standard sentence logic is non-contradictory when only a finite non-null number of *SSL* formulas are interpreted as concrete propositions with a well defined truth value, necessarily some ones as true, some ones as false. A consequence of this result is that all the mathematical propositions that contain numbers with infinite digits are contradictory. This fact has relevant implications to clarify the relations between Matte Blanco's bi-logic and the ordinary numerical calculus.

2. Information Entropy

The information entropy H is the measure in *bit* of the information which any message contains and which is defined in this way:

$$H = -\sum_{h=1}^{n} p_h \log_2 \left(p_h \right)$$

where n is the number of possible events which the message could describe, p_h the probability to obtain the h-th possible event. We clarify by two examples:

Example 2.1

Consider the money throwing. We want calculus the entropy which the knowledge of its result contains. There are only 2 possible events (heads or tails?) and they have the same probability equal to $\frac{1}{2}$. Therefore we have:

$$\begin{aligned} H &= -\left(\tfrac{1}{2} \log_2 \left(\tfrac{1}{2} \right) + \tfrac{1}{2} \log_2 \left(\tfrac{1}{2} \right) \right) = \\ &= -\left(2 \cdot \frac{1}{2} \log_2 \left(\frac{1}{2} \right) \right) = \\ &= -\left(2 \cdot \frac{1}{2} \cdot -1 \right) = 1 \; bit \end{aligned}$$

Therefore the information unit (the *bit*) is equal to knowledge of which event happens between two possible events which have the same probability.

Example 2.2

Consider the throwing of a die. We want calculus the entropy that the knowledge of its result contains. There are only 6 possible events (1, 2, 3, 4, 5, 6) and they have the same probability equal to $\frac{1}{6}$. Therefore we have:

$$H = -\left(6 \cdot \frac{1}{6} \log_2 \left(\frac{1}{6} \right) \right) \cong 2.58$$

If the possible events that a message can describe have not the same probability, then the information of this message is less than when these possible events have the same probability. We clarify by an example:

Example 2.3

The knowledge of which between two possible events of equal probability happens is equal to 1 *bit* of information. If the two events have not the same probability, but the probability of the former is 0.7 and the probability of the latter is 0.3, then $H = -\left(0.7 \log_2(0.7) + 0.3 \log_2(0.3)\right) \cong 0.8813$ which is less than 1.

If the possible event that a message can describe is one (or if all the other events have probability equal to zero), then the entropy of this message is zero.

3. Bi-logic

The fundamental characteristics of the psychiatric diseases are explained by the Matte Blanco's symmetry principle which we formulate in this way: in the psychiatric diseases some asymmetric relations become symmetric (e. g. ‘Peter drinks the water' becomes equivalent to ‘The water drinks Peter') and sometimes the asymmetric relation ‘belongs to' becomes the symmetric relation ‘is equal to' (e. g. a psychotic thought is: ‘Peter is a boy', ‘I am a boy', hence ‘I am Peter', i. e. the propositions ‘Peter is a boy' = ‘Peter belongs to the boy set' and ‘I am a boy' = ‘I belong to the boy set' become ‘Peter is equal to the boy set' and ‘I am equal to the boy set' respectively by the symmetry principle and permit the conclusion ‘I am Peter').

If we apply the symmetry principle to all the relations of any algebraic structure, the this one becomes a set with only an

element on which only the identity relation is put: all the distinct elements of the algebraic structure are considered identical, all the distinct relations of the algebraic structure are considered equal to identity relations.

Consider a partial application of the symmetry principle in any algebraic structure. Only some distinct elements become equal among them and only some asymmetric relations become symmetric. In according with Matte Blanco call any algebraic structure with a partial application of the symmetry principle *bi-logic* structure and call *bi-logic* the general theory of the bi-logic structures. The most typical bi-logic structure is the freudian unconscious, but we can individuate also other bi-logic structures. We give an example:

Example 3.1:

An increase of the calculus inexactness in any computer provokes the unification of distinct numbers in it (e. g. if we go from the three decimal digit approximation to two decimal digit approximation, then the distinct numbers 0.312 and 0.314 become both 0.31). Therefore we can obtain the calculus with two decimal digit approximation by partial application of the symmetry principle on the calculus with three decimal digit approximation. In general a calculus with n decimal digit approximation is a bi-logical structure which is obtained by partial application of the symmetry principle to the calculus with n+1 decimal digit. Hence every numeric calculus with a finite number of decimal digit of approximation is a bi-logical structure.

4. Axiomatic Formal Theories

An *axiomatic formal theory* **S** is defined when the following condictions are satisfied:

1) There is an at most countable set of **S** symbols *SY*(S).

2) There is the set of the finite sequence of **S** symbols *EX* (S) which we call **S** *expression set*.

3) There is a subset $WFF(\mathrm{S})$ of $EX(\mathrm{S})$ which we call **S** *well formed formula set*.

4) There is an algorithm $AWFF(\mathrm{S})$ such that if we apply it to a **S** expression [an element of $EX(\mathrm{S})$], then its result shows if this expression belongs to $WFF(\mathrm{S})$.

5) There is a subset $AX(\mathrm{S})$ of $WFF(\mathrm{S})$ which we call **S** *axiom set*.

6) There is an algoritm $AAX(\mathrm{S})$ such that if we apply it to a **S** well formed formula [an element of $WFF(\mathrm{S})$], then its result shows if this expression belongs to $AX(\mathrm{S})$.

7) There is a finite set of relations $INFER(\mathrm{S})$ in which for every its element R_k there is a positive integer j such that, for every set of j well formed formulas (abbreviate wffs) SW and for every well formed formula (abbreviate wff) $\mathcal{A}_k$, we can know if SW and $\mathcal{A}_k$ are in relation R_k always.

A **S** *demonstration* (or **S** *proof*) is a finite sequence of **S** wffs such that every its wff or is a **S** axiom or is in relation R_k with a set of wffs which precede in the sequence, where R_k is an elemente of $INFER(\mathrm{S})$.

A **S** *theorem* is the last formula of a **S** *demonstration*.

A **S** *demonstration* of a wff $\mathcal{A}_k$ from a set $ANTEC$ of wffs $\{\mathcal{A}_0, \ldots, \mathcal{A}_n\}$ is a finite sequence of **S** wffs such that its last wff is $\mathcal{A}_k$ and every other its wff or is a **S** axiom, or it is in relation R_k with a set of wffs which precede in the sequence, where R_k

is an elemente of *INFER*(S), or it is an element of *ANTEC*. If there is a prof of a wff $\mathcal{A}_k$ from a set *ANTEC* of wffs $\{\mathcal{A}_0, \ldots, \mathcal{A}_n\}$, then we call $\mathcal{A}_k$ *SSL derivation* of $\{\mathcal{A}_0, \ldots, \mathcal{A}_n\}$ and we write $\{\mathcal{A}_0, \ldots, \mathcal{A}_n\} \mapsto \mathcal{A}_k$; in patricular, if $\mathcal{A}_k$ is a **S** *theorem* or a **S** *axiom*, then we write $\varnothing \mapsto \mathcal{A}_k$ or, shortly, $\mapsto \mathcal{A}_k$.

5. Standard Sentence Logic (*SSL*) as Formal Theory

Define an axiomatic formal theory **P** such that:

1) $SY(\mathbf{P}) = \{\neg, \ldots, \mathcal{A}_0, \ldots, \mathcal{A}_k, \ldots, (,)\}$

2) The algoritm $AWFF(\mathbf{P})$ is so defined:

2a) $\mathcal{A}_0, \ldots, \mathcal{A}_k$ are wffs;

2b) if $\mathcal{A}$ is a wff, then $\neg\mathcal{A}$ is a wff;

2c) if $\mathcal{A}$ and $\mathcal{B}$ are wffs, then $(\mathcal{A} \ldots \mathcal{B})$ is a wff;

2d) an expression of **P** is a wff if and only if we can apply to it the case 2a) or the case 2b) or the case 2c).

3) The algoritm $AAX(\mathbf{P})$ is so defined:

if $\mathcal{A}$, $\mathcal{B}$, C are wffs of **P**, then the following expressions are axioms of **P**:

3') $(\mathcal{A}\supset(\mathcal{B}\supset\mathcal{A}))$;

3") $((\mathcal{A}\supset(\mathcal{B}\supset C))\supset((\mathcal{A}\supset\mathcal{B})\supset(\mathcal{A}\supset C)))$;

3''') $((\neg\mathcal{B}\supset\neg\mathcal{A})\supset((\neg\mathcal{B}\supset\mathcal{A})\supset\mathcal{B}))$.

4) The only relation which belongs to $INFER(\mathbf{P})$ is the following: $\{\mathcal{A}\supset\mathcal{B}, \mathcal{A} \mapsto \mathcal{B}\}$ where $\mathcal{A}$ and $\mathcal{B}$ are wffs.

If $\mathcal{A}$ and $\mathcal{B}$ are wffs, then put in **P** the following abreviations:

*) $(\mathcal{A}\wedge\mathcal{B})$ abbreviates $\neg(\mathcal{A}\supset\neg\mathcal{B})$;

**) $(\mathcal{A}\vee\mathcal{B})$ abbreviates $(\neg\mathcal{A})\supset\mathcal{B}$;

***) $(\mathcal{A}\equiv\mathcal{B})$ abbreviates $((\mathcal{A}\supset\mathcal{B})\wedge(\mathcal{B}\supset\mathcal{A}))$.

Call the symbols $\neg$, $\supset$, $\wedge$, $\vee$, $\equiv$ *connectives* and the symbols $\mathcal{A}_0, \ldots, \mathcal{A}_k, \ldots$ *atomic sentences*.

Interpretate $\mathcal{A}_0, \ldots, \mathcal{A}_k, \ldots$as *SSL* atomic sentences, $\neg$ as *not*, $\supset$ as *if …, then …* , $\wedge$ as *and*, $\vee$ as *inclusuve or*, $\equiv$ as *… if and only if …* in *SSL*. We obtain a complete representation of the standard sentence logic *SSL* [Mendelson, 1964] .

6. *SSL* as Interpreted Boole's Algebra

Let $B = \{x, y, z, \ldots\}$ a non-empty set, f a function from B to B, and g a function from B^2 to B. Write x' instead of $f(x)$, $x \cap y$ instead of $g(x, y)$.

$\langle B, f, g \rangle$ is a *Boole's algebra* if and only if the following provisos are satisfied for every x, y, z in B :

1a) $x \cap y = y \cap x$,
2a) $(x \leftrightarrow y) \cap z = x \cap (y \leftrightarrow z)$,
3a) $x \cap x' = z \cap z'$ iff $x \cap y = x$.

Put the following abbreviations for every x, y in B :

1b) $x \cup y =_{df} x' \cap y'$,
2b) $x \leq y =_{df} x \cap y = x$.

We let proof to the reader that for every x, y in B :

1c) $x \cap x' = y \cap y'$.

Put in general:

1d) $x \cap x' = 0$,
2d) $1 = 0'$.

We let proof to the reader that for every x, y in B :

1e) $x \cup x' = 1$,
2e) $\leq$ is a reflexive partial order on B,
3e) $<B, f, \approx>$ is a Boole's algebra.

Define *ideal* in $<B, f, g>$ a non-empty subset J of B such that:

1f) $x \in J$ and $y \in J$ imply $x \cup y \in J$,
2f) $x \in J$ and $y \in B$ imply $x \cap y \in J$.

We let proof to the reader that $\{0\}$ and B are ideals. Precisely $\{0\}$ is a *trivial ideal*, and B is an *improper ideal*. An ideal distinct from B is called *proper*.

A proper ideal is *maximal* when it is not included in an other proper ideal. We let proof to the reader that for every x in B :

1g) J is a maximal ideal iff $x \in J$ or $x' \in J$,
2g) Every proper ideal is included in a maximal ideal (consider the choice axiom …).

If we put a correspondence between the wffs of **P** and the elements of $<B, f, g>$ in way that:

1h) $\mathcal{A} \leftrightarrow x$ iff $\neg\mathcal{A} \leftrightarrow x'$,

2h) $\mathcal{A} \leftrightarrow x$ and $\mathcal{A} \leftrightarrow y$ iff $\mathcal{A} \wedge \mathcal{B} \leftrightarrow x \cap y$,

then we have that:

1i) $\mapsto \mathcal{A}$ iff $\mathcal{A} \leftrightarrow 1$,

2i) $\mapsto \neg\mathcal{A}$ iff $\mathcal{A} \leftrightarrow 0$;

This result is a variation of the Kalmàr's completeness theorem. It shown that *SSL* is a Boole's algebra and therefore it permitts us to use the standard Boolean calcule to proof the **P** theorems. To use it we chose the following rules:

11) we represent in the tautology calculus any atomic sentence $\mathcal{A}_n$ which is put in an other sentence with m and only m distinct atomic sentences with a Boolean function (the $\mathcal{A}_n$ *truth table*) such that:

$$\left.\begin{matrix} \left.\begin{matrix} 1 \\ \cdots \\ 1 \end{matrix}\right\} 2^n \text{ times} \\ \left.\begin{matrix} 0 \\ \cdots \\ 0 \end{matrix}\right\} 2^n \text{ times} \end{matrix}\right\} \frac{2^m}{2^{n+1}} \text{ times,}$$

Example 1

$\mathcal{A}_0$ is a *SSL* wff with 1 distinct atomic sentence $\mathcal{A}_0$. So $\mathcal{A}_0$ represents

$$\left.\begin{matrix} \left.\begin{matrix} 1 \\ \cdots \\ 1 \end{matrix}\right\} 2^0 \text{ times} \\ \left.\begin{matrix} 0 \\ \cdots \\ 0 \end{matrix}\right\} 2^0 \text{ times} \end{matrix}\right\} \frac{2^1}{2^{0+1}} \text{ times} = \begin{matrix} 1 \\ 0 \end{matrix} .$$

Example 2

$\mathcal{A}_0 \wedge \mathcal{A}_1$ is a *SSL* wff with 2 distinct atomic sentences $\mathcal{A}_0$, $\mathcal{A}_1$. So $\mathcal{A}_0$ represents $\left.\begin{matrix} \left.\begin{matrix} 1 \\ \cdots \\ 1 \end{matrix}\right\} 2^0 \text{ times} \\ \left.\begin{matrix} 0 \\ \cdots \\ 0 \end{matrix}\right\} 2^0 \text{ times} \end{matrix}\right\} \frac{2^2}{2^{0+1}} \text{ times} = \begin{matrix} 1 \\ 0 \\ 1 \\ 0 \end{matrix}$ and $\mathcal{A}_1$ represents

$$\left.\begin{array}{l}\left.\begin{array}{c}1\\ \cdots\\ 1\end{array}\right\}2^{1}\text{ times}\\ \left.\begin{array}{c}0\\ \cdots\\ 0\end{array}\right\}2^{1}\text{ times}\end{array}\right\}\frac{2^{2}}{2^{1+1}}\text{ times} = \begin{array}{c}1\\1\\0\\0\end{array};$$

2l) we represent every *SSL* connective by a Boolean operation, in particular we put:

- *and* (∧) corresponds to …∩… and:

$$\begin{aligned} 1 \wedge 1 &= 1 \\ 1 \wedge 0 &= 0 \\ 0 \wedge 1 &= 0, \\ 0 \wedge 0 &= 0 \end{aligned}$$

- *or inclusuve* (∨) corresponds to …∪… and:

$$\begin{aligned} 1 \vee 1 &= 1 \\ 1 \vee 0 &= 1 \\ 0 \vee 1 &= 1, \\ 0 \vee 0 &= 0 \end{aligned}$$

- *implies* (⊃) corresponds to …' ∪… and:

$$\begin{aligned} 1 \supset 1 &= 1 \\ 1 \supset 0 &= 0 \\ 0 \supset 1 &= 1, \\ 0 \supset 0 &= 1 \end{aligned}$$

- *not* (¬) corresponds to …' and:

$$\begin{aligned} \neg 1 &= 0 \\ \neg 0 &= 1, \end{aligned}$$

- *if and only if* (≡) corresponds to (…′∪…)∩(…∪…′) and:

$$\begin{array}{lcl} 1 \equiv 1 & = & 1 \\ 1 \equiv 0 & = & 0 \\ 0 \equiv 1 & = & 0 \\ 0 \equiv 0 & = & 1 \end{array}.$$

In the tautology calcule for every *SSL* theorem $\vdash \mathcal{A}$, $\mathcal{A}$ corresponds always to a Boolean function which contains only the Boolean value *1* , so we call $\mathcal{A}$ *tautology* ; instead if $\vdash \neg\mathcal{A}$, then $\mathcal{A}$ corresponds always to a Boolean function which contains only the Boolean value *0*, so we call $\mathcal{A}$ *contradiction*. If a *SSL* wff is neither a tautology nor a contradiction (its correspondent Boolean functions contains either some occurrences of *1* or of *0*), then it is a *neuter formula*.

7. Informatic entropy in Standard Sentence Logic

The tautology calculus permits us to evaluate the information entropy of the *SSL* wffs. In fact a determined Boolean function, in which there are occurrences of *1* and/or of *0*, corresponds to every wff. We can calculate the information entropy $\mathrm{H}(\mathcal{A})$ of a given wff $\mathcal{A}$ by calculating the probabilities of *0* and of *1* to occur in its correspondent truth table. We clarify by some examples:

Example 7.1

Every atomic sentence $\mathcal{A}_n$ corresponds to truth table $\begin{array}{l} \left.\begin{array}{c} 1 \\ \cdots \\ 1 \end{array}\right\} 2^n \text{ times} \\ \left.\begin{array}{c} 0 \\ \cdots \\ 0 \end{array}\right\} 2^n \text{ times} \end{array}$ in which *0* and *1* are the only possible values and they have the same number of occurrences. It is the same case of the throwing of a money. Therefore:

$$H(\mathcal{A}_n) = -\left(\frac{1}{2}\log_2\left(\frac{1}{2}\right)+\frac{1}{2}\log_2\left(\frac{1}{2}\right)\right) = -\left(2\cdot\frac{1}{2}\log_2\left(\frac{1}{2}\right)\right) = -\left(2\cdot\frac{1}{2}\cdot -1\right) = 1\ bit.$$

Example 7.2

$$H(\mathcal{A}_1\wedge\mathcal{A}_0) = H\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = -\left(\frac{1}{4}\log_2\left(\frac{1}{4}\right)+\frac{3}{4}\log_2\left(\frac{3}{4}\right)\right) \cong -\left(\frac{1}{4}\cdot -2+\frac{3}{4}\cdot -0.415\right) \cong 0.811\ bit.$$

Example 7.3

$$H(\mathcal{A}_1\vee\mathcal{A}_0) = H\begin{pmatrix}1\\1\\1\\0\end{pmatrix} = -\left(\frac{3}{4}\log_2\left(\frac{3}{4}\right)+\frac{1}{4}\log_2\left(\frac{1}{4}\right)\right) \cong -\left(\frac{3}{4}\cdot -0.415+\frac{1}{4}\cdot -2\right) \cong 0.811\ bit.$$

Example 7.4

$$H(\mathcal{A}_1\equiv\mathcal{A}_0) = H\begin{pmatrix}1\\0\\0\\1\end{pmatrix} = -\left(\frac{1}{2}\log_2\left(\frac{1}{2}\right)+\frac{1}{2}\log_2\left(\frac{1}{2}\right)\right) = -\left(2\cdot\frac{1}{2}\log_2\left(\frac{1}{2}\right)\right) = -\left(2\cdot\frac{1}{2}\cdot -1\right) = 1\ bit.$$

Example 7.5

$$H(\mathcal{A}_0\wedge\neg\mathcal{A}_0) = H\begin{pmatrix}0\\0\\0\\0\end{pmatrix} = -\left(\frac{0}{4}\log_2\left(\frac{0}{4}\right)+\frac{4}{4}\log_2\left(\frac{4}{4}\right)\right) \cong -\left(0\cdot -+1\cdot -0\right) = 0\ bit.$$

Example 7.6

$$H(\mathcal{A}_0\vee\neg\mathcal{A}_0) = H\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = -\left(\frac{4}{4}\log_2\left(\frac{4}{4}\right)+\frac{0}{4}\log_2\left(\frac{0}{4}\right)\right) \cong -\left(1\cdot -0+0\cdot -\right) = 0\ bit.$$

The last two examples show clearly that $H(\mathcal{A}) \neq 0$ iff $\mathcal{A}$ is a

neuter formula. We can think that $H(\mathcal{A})$ is a measure of the 'neutrality' of the wff $\mathcal{A}$. This 'neutrality' is null when $\mathcal{A}$ is a tautology (i. e. a *SSL* theorem) or $\mathcal{A}$ is a contradiction (i. e. the negation of a *SSL* theorem). $0 < H(\mathcal{A}) \leq 1$ iff $\mathcal{A}$ is a neuter formula. If $\mathcal{A}$ is an atomic formula, then $H(\mathcal{A}) = 1$, but also the entropy of some molecular formulas is equal to 1.

8. Informatic entropy and polyadic connectives

Let $\mathcal{A}(k)$ (where k is any positive integer number) be any *SSL* wff. $\bigwedge_{i=0}^{n} \mathcal{A}(i)$ is the logical product of n *SSL* wffs and $\bigvee_{i=0}^{n} \mathcal{A}(i)$ is the logical sum of n *SSL* wffs. We will proof that:

Theorem 8.1: *The logical product of infinite formulas non-equivalent among them and the logical sum of infinite formulas non-equivalent among them are not neuter formulas*, i. e.:

$$\lim_{n\to\infty} H\left(\bigwedge_{i=0}^{n} \mathcal{A}(i)\right) = \lim_{n\to\infty} H\left(\bigvee_{i=0}^{n} \mathcal{A}(i)\right) = 0$$

<u>Proof:</u>

By definition of $H(\mathrm{A})$ we observe that if $m > n$, then $H\left(\bigwedge_{i=0}^{n} \mathcal{A}(i)\right) > H\left(\bigwedge_{i=0}^{m} \mathcal{A}(i)\right)$ and $H\left(\bigvee_{i=0}^{n} \mathcal{A}(i)\right) > H\left(\bigvee_{i=0}^{m} \mathcal{A}(i)\right)$. As for every positive number small at will ε there is a natural number n such that $H\left(\bigwedge_{i=0}^{n} \mathcal{A}(i)\right) < \varepsilon$ and $H\left(\bigvee_{i=0}^{n} \mathcal{A}(i)\right) < \varepsilon$, then, for a

known rule of mathematical analysis, we deduce our theorem.

Now we can understand if $\lim_{n\to\infty} H\left(\bigwedge_{i=0}^{n} \mathcal{A}(i)\right)$ and $\lim_{n\to\infty} H\left(\bigvee_{i=0}^{n} \mathcal{A}(i)\right)$ are tautologies or contradictions (i. e. if they are *SSL* theorems or negations of *SSL* theorems).

In general call $\mathbb{O}(\mathcal{A})$ the ratio $\frac{\text{number of } 0}{\text{number of } 1}$ of the truth table of $\mathcal{A}$. It is evident that $\mathbb{O}(\mathcal{A}) = \infty$ iff $\mathcal{A}$ is a contradiction and $\mathbb{O}(\mathcal{A}) = 0$ iff $\mathcal{A}$ is a tautology. Put $(\mathbb{O}(\mathcal{A}))^{-1} = \mathbb{1}(\mathcal{A})$.

Consider $\bigwedge_{i=0}^{n} \mathcal{A}(i)$. In its truth table the increase of n provokes the increase of the ratio $\frac{\text{number of } 0}{\text{number of } 1}$. As for every positive number great at will ε there is a positive integer number n such that $\mathbb{O}\left(\bigwedge_{i=0}^{n} \mathcal{A}(i)\right) > \varepsilon$, then, for a known rule of mathematical analysis, we deduce $\lim_{n\to\infty} \mathbb{O}\left(\bigwedge_{i=0}^{n} \mathcal{A}(i)\right) = \infty$, we can affirm that:

Theorem 8.2: *The logical product of infinite non-equivalent among them formulas is a contradiction.*

Consider $\bigvee_{i=0}^{n} \mathcal{A}(i)$ also. In its truth table the increase of n provokes the decrease of the ratio $\frac{\text{number of } 0}{\text{number of } 1}$ .As for every positive number small at will ε there are a positive integer number n such that $\mathbb{O}\left(\bigvee_{i=0}^{n} \mathcal{A}(i)\right)<\varepsilon$, then, for a known rule of mathematical analysis, we deduce $\lim_{n\to\infty} \mathbb{O}\left(\bigvee_{i=0}^{n} \mathcal{A}(i)\right) = 0$, we can affirm that:

Theorem 8.3: *The logical sum of infinite non-equivalent among them formulas is a tautology.*

If we accept the Karnap's interpretation of the modal connectives (necessity = tautology) , we can affirm also:

8a) if $\Box\mathcal{A}$ means '$\mathcal{A}$ is necessary', then $\Box\mathcal{A}$ abbreviates $\mathbb{O}(\mathcal{A}) = 0$;

8b) if $\Diamond\mathcal{A}$ means '$\mathcal{A}$ is possible (necessary or contingent), then $\Diamond\mathcal{A}$ abbreviates $\neg\Box\neg\mathcal{A}$;

8c) if $\mathrm{O}\mathcal{A}$ means '$\mathcal{A}$ is contingent', then $\mathrm{O}\mathcal{A}$ is the abreviation of $\neg\Box\mathcal{A} \wedge \neg\Box\neg\mathcal{A}$.

We let to reader the various relations which exists among the functions H (…), $\mathbb{O}$(…), $\mathbb{1}$(…) and the connectives $\Box$…,

◊…, O… .

Consider Malatesta's polyadic connectives:

8d) *Sequence implication* : $\mathop{C}_{i=0}^{n}{}_{S}\,\mathcal{A}(i)$. It is equal to:

$\mathcal{A}(0)\supset(\mathcal{A}(1)\supset(\mathcal{A}(2)\supset(\ldots\supset(\mathcal{A}(n\text{-}1)\supset\mathcal{A}(n))\ldots)))$;

8e) *Chain implication* : $\mathop{C}_{i=0}^{n}{}_{C}\,\mathcal{A}(i)$. It is equal to: $\mathcal{A}(0)\supset\mathcal{A}(1))\wedge(\mathcal{A}(1)\supset\mathcal{A}(2))\wedge\ldots\wedge(\mathcal{A}(n\text{-}1)\supset\mathcal{A}(n)))$;

8f) *Converse sequence implication*: $\mathop{B}_{i=0}^{n}{}_{S}\,\mathcal{A}(i)$. It is equal to: $\mathcal{A}(n)\supset(\mathcal{A}(n\text{-}1)\supset(\ldots\supset(\mathcal{A}(2)\supset(\mathcal{A}(1))\ldots))$;

8g) *Converse chain implication*: $\mathop{B}_{i=0}^{n}{}_{C}\,\mathcal{A}(i)$. It is equal to: $(\mathcal{A}(n)\supset\mathcal{A}(n\text{-}1))\wedge\ldots\wedge(\mathcal{A}(2)\supset\mathcal{A}(1))\wedge(\mathcal{A}(1)\supset\mathcal{A}(0))$;

8h) *Polyadic equivalence*: $\mathop{E}_{i=0}^{n}\,\mathcal{A}(i)$. It is equal to: $(\mathcal{A}(0)\equiv\mathcal{A}(1))\wedge(\mathcal{A}(1)\equiv\mathcal{A}(2))\wedge\ldots\wedge(\mathcal{A}(n\text{-}1)\equiv\mathcal{A}(n))$.

We can affirm the following theorems.

Theorem 8.4: $\lim_{n\to\infty}\mathop{C}_{i=0}^{n}{}_{S}\,\mathcal{A}(i)$ *is a tautology.*

<u>Proof:</u>

$\mathcal{A}(0)\supset(\mathcal{A}(1)\supset(\mathcal{A}(2)\supset(\ldots\supset(\mathcal{A}(n\text{-}1)\supset\mathcal{A}(n))\ldots)))$ =

$$\neg\mathcal{A}(0)\vee(\neg\mathcal{A}(1)\vee(\neg\mathcal{A}(2)\vee(\ldots\vee(\neg\mathcal{A}(n\text{-}1)\vee\mathcal{A}(n))\ldots))) =$$
$$\neg\mathcal{A}(0)\vee\neg\mathcal{A}(1)\vee\neg\mathcal{A}(2)\vee\ldots\vee\neg\mathcal{A}(n\text{-}1)\vee\mathcal{A}(n)$$

therefore:

$$\lim_{n\to\infty}\mathbb{O}\left(\mathop{C_S}_{i=0}^{n}\mathcal{A}(i)\right) = \lim_{n\to\infty}\mathbb{O}\left(\bigvee_{i=0}^{n}\neg\mathcal{A}(i)\right);$$

hence, for the theorem 8.3, this theorem is true, *c. d. d.*

If $\mathcal{A}\,|\,\mathcal{B} = \neg\mathcal{A}\vee\neg\mathcal{B}$, then $|$ is the *Sheffer's connective*.

Define the polyadic Sheffer's connective $\mathop{|}_{i=0}^{n}\mathcal{A}(i)$ as $\neg\mathcal{A}(0)\vee\neg\mathcal{A}(1)\vee\neg\mathcal{A}(2)\vee\ldots\vee\neg\mathcal{A}(n\text{-}1)\vee\mathcal{A}(n)$. So:

Theorem 8.5: $\lim_{n\to\infty}\mathbb{O}\left(\mathop{C_S}_{i=0}^{n}\mathcal{A}(i)\right) = \lim_{n\to\infty}\mathbb{O}\left(\mathop{|}_{i=0}^{n}\mathcal{A}(i)\right).$

Theorem 8.6: $\lim_{n\to\infty}\mathop{C_C}_{i=0}^{n}\mathcal{A}(i)$ *is a contradiction.*

<u>Proof:</u>

$$(\mathcal{A}(0)\supset\mathcal{A}(1))\wedge(\mathcal{A}(1)\supset\mathcal{A}(2))\wedge\ldots\wedge(\mathcal{A}(n\text{-}1)\supset\mathcal{A}(n))$$

therefore:

$$\lim_{n\to\infty}\mathbb{O}\left(\mathop{C_C}_{i=0}^{n}\mathcal{A}(i)\right) = \lim_{n\to\infty}\mathbb{O}\left(\bigwedge_{i=0}^{n-1}(\mathcal{A}(i)\supset\mathcal{A}(i+1))\right);$$

hence, for the theorem 8.2, this theorem is true, *c. d. d.*

Theorem 8.7: $\lim_{n\to\infty} \mathop{B}_{i=0}^{n}{}_{S}\, \mathcal{A}(i)$ *is a tautology.*

<u>Proof:</u>

$$\mathcal{A}(n)\supset(\mathcal{A}(n\text{-}1)\supset(\ldots\supset(\mathcal{A}(2)\supset(\mathcal{A}(1)\supset\mathcal{A}(0)))\ldots)) =$$

$$\neg\mathcal{A}(n)\vee(\neg\mathcal{A}(n\text{-}1)\vee(\ldots\vee(\neg\mathcal{A}(2)\vee(\neg\mathcal{A}(1)\vee\mathcal{A}(0)))\ldots)) =$$

$$\neg\mathcal{A}(n)\vee\neg\mathcal{A}(n\text{-}1)\vee\ldots\vee\neg\mathcal{A}(2)\vee\neg\mathcal{A}(1)\vee\mathcal{A}(0)\ldots$$

therefore:

$$\lim_{n\to\infty}\mathbb{O}\left(\mathop{B}_{i=0}^{n}{}_{S}\, \mathcal{A}(i)\right) = \lim_{n\to\infty}\mathbb{O}\left(\bigvee_{i=n}^{0}\neg\mathcal{A}(i)\right);$$

hence, for the theorem 8.3, this theorem is true, *c. d. d.*

Theorem 8.8: $\lim_{n\to\infty} \mathop{B}_{i=0}^{n}{}_{C}\, \mathcal{A}(i)$ *is a contradiction.*

<u>Proof:</u>

$$(\mathcal{A}(n)\supset\mathcal{A}(n\text{-}1))\wedge\ldots\wedge(\mathcal{A}(2)\supset\mathcal{A}(1))\wedge(\mathcal{A}(1)\supset\mathcal{A}(0))$$

therefore:

$$\lim_{n\to\infty}\mathbb{O}\left(\mathop{B}_{i=0}^{n}{}_{C}\, \mathcal{A}(i)\right) = \lim_{n\to\infty}\mathbb{O}\left(\bigwedge_{i=n}^{1}(\mathcal{A}(i)\supset\mathcal{A}(i-1))\right);$$

hence, for the theorem 8.2, this theorem is true, *c. d. d.*

Theorem 8.9: $\lim_{n\to\infty} \mathop{E}_{i=0}^{n}{}_{C}\, \mathcal{A}(i)$ *is a contradiction.*

<u>Proof:</u>

$$(\mathcal{A}(0)\equiv\mathcal{A}(1))\wedge(\mathcal{A}(1)\equiv\mathcal{A}(2))\wedge\ldots\wedge(\mathcal{A}(n\text{-}1)\equiv\mathcal{A}(n)) =$$

$$(\mathcal{A}(0)\supset\mathcal{A}(1))\wedge(\mathcal{A}(1)\supset\mathcal{A}(2))\wedge\ldots\wedge(\mathcal{A}(n\text{-}1)\supset\mathcal{A}(n))\wedge(\mathcal{A}(n)\supset\mathcal{A}(n\text{-}1))\wedge\ldots\wedge(\mathcal{A}(2)\supset\mathcal{A}(1))\wedge(\mathcal{A}(1)\supset\mathcal{A}(0))$$

therefore:

$$\lim_{n\to\infty}\mathbb{O}\left(\mathop{E}_{i=0}^{n}\mathcal{A}(i)\right)=$$

$$\lim_{n\to\infty}\mathbb{O}\left(\bigwedge_{i=0}^{n-1}(\mathcal{A}(i)\supset\mathcal{A}(i+1))\wedge(\mathcal{A}(i+1)\supset\mathcal{A}(i))\right);$$

hence, for the theorem 8.2, this theorem is true, *c. d. d.*

Semantically, the theorems 8.2 and 8.3 have an important consequence. It is known that *SSL* is interpretable by substitution of some its atomic sentences with concrete propositions which have a well defined truth value (true or false). Consider now an interpretation of *n SSL* wffs $\mathcal{B}(1)$, …, $\mathcal{B}(n)$. If the proposition that corresponds to $\underset{1\le k\le n}{\mathcal{B}(k)}$ is true, then put $\underset{1\le k\le n}{\mathcal{B}(k)}\equiv\underset{1\le k\le n}{\mathcal{A}(k)}$. Instead, if the proposition which corresponds to $\underset{1\le k\le n}{\mathcal{B}(k)}$ is false, then put $\neg\underset{1\le k\le n}{\mathcal{B}(k)}\equiv\underset{1\le k\le n}{\mathcal{A}(k)}$. In this way every attribution of *n* propositions to *n* *SSL* wffs affirms the formula: $\bigwedge_{i=0}^{n}\mathcal{A}(i)$. If $n=\infty$, then $\bigwedge_{i=0}^{n}\mathcal{A}(i)$ is a contradiction for the theorem 8.2.On the other side, $\lim_{n\to\infty}\bigvee_{i=0}^{n}\mathcal{A}(i)$ is a tautology for theorem 8.3. Therefore every non-contradictory interpretation of *SSL* must be contain some *SSL* sentences which are interpreted as concrete propositions with a well defined truth value (true or false), i. e. it is necessary that $n>0$.

Hence, we can affirm the following meta-theorem:

Meta-theorem 8.10: *Any interpretation of the standard sentence logic is non-contradictory when only a finite non-null number of SSL formulas are interpreted as concrete propositions with a well defined truth value, necessarily some ones as true, some ones as false.*

9. Numeric Calculus as Bi-logic Structure

Meta-theorem 9.1: *Any mathematical proposition is non-contradictory when all the eventual numbers which are in it are done by a finite number of digits.*

Proof:

Consider any mathematical identity: $e = n$, were e is any mathematical expression and n is a number at will. Let $b(n)$ be the binary representation of n. Every digit of $b(n)$ is 1 or 0.

Consider now the following propositions where we consider as first digit the more right digit:

$\mathcal{B}(1)$ = 'e is equal to a binary number the whose 1-st digit is 1',

$\mathcal{B}(2)$ = 'e is equal to a binary number the whose 2-nd digit is 1',

$\mathcal{B}(3)$ = 'e is equal to a binary number the whose 3-rd digit is 1',

……

$\mathcal{B}(n)$ = 'e is equal to a binary number the whose n-th digit is 1',

……

If $\mathcal{B}(n)$ is false, then it is evident that ''e is equal to a binary number the whose n-th digit is 0'. In this way every identity as $e = b(n)$ can be represented with a logical product of sentences $\mathcal{B}(n)$ and/or of their negations. We clarify by an example:

Example 9.1

Consider the identity $e = 5$. The usual binary representation of 5 is 101. Therefore we can write $e = 101$ by substitution of the base 10 with the base 2 in the numbers of our identity. But $e = 101$ can be represented by $\mathcal{B}(n)$ sentence in this way:

'e is equal to a binary number the whose 1-st digit is 1' and
'e is equal to a binary number the whose 2-nd digit is 0' and
'e is equal to a binary number the whose 3-rd digit is 1'.

This sentence becomes: $\mathcal{B}(3) \wedge \neg\mathcal{B}(2) \wedge \mathcal{B}(1)$.

Observe that in the common numerical calculus 5 is the abbreviation of …….00005 In effect in the standard mathematics the number of digits that every number contains is potentially infinite and only the calculator limits oblige us to consider only the more right k digits. If the considered numbers terminate with infinite 0, then our conventions make seem us that the calculus happens on numbers with a finite number of digits. In effect the true representation of $e = 5$ in terms of $\mathcal{B}(n)$ sentences is:

$$\lim_{n\to\infty} \bigwedge_{i=n}^{4} \neg\mathcal{B}(i) \wedge \mathcal{B}(3) \wedge \neg\mathcal{B}(2) \wedge \mathcal{B}(1)$$

Put $\mathcal{A}(n)$ equal to $\mathcal{B}(n)$ when $\mathcal{B}(n)$ is true and $\mathcal{A}(n)$ equal to $\neg\mathcal{B}(n)$ when $\mathcal{B}(n)$ is false. Then, we can represent every mathematical identity $e = m$, where e is any mathematical expression and m is any number, with the sentence: $\bigwedge_{i=1}^{n} \mathcal{A}(i)$ where n represents the number of digits used to represent the numbers in $e = m$. If these numbers are infinitely precise (i. e. if they are represented with an infinite number of digits), then the representation of $e = n$ becomes $\lim_{n\to\infty} \bigwedge_{i=1}^{n} \mathcal{A}(i)$, i.e. a

contradiction. If the infinite precision of the numbers (i. e. their representation with infinite digits) implies the representation of the mathematical identities that contain them with a contradictory sentence, then we can consider only the number with a finite precision (i. e. which are represented with a finite number of digits) to eliminate some possible contradictions, *c. d. d.*

Consider another time the example 3.1 in the paragraph 3 on the bi-logic. It shows as a calculus, in which the numbers are finitely precise (i. e. they are represented with n [where n is finite] digits), is a bi-logical structure which is obtained by partial application of the symmetry principle to the correspondent calculus in which the number are represented with n+1digits. As every non-contradictory calculus must happen on finitely precise numbers for the theorem 9.2, then every non-contradictory numeric calculus must be a bi-logical structure. Therefore we can affirm that:

Meta-theorem 9.2: *Every numeric calculus must be a bi-logical structure.*

10. References

Bell J. L. Machover L.
A Course of Mathematical Logic
North-Holland, Amsterdam - New York - Oxford, 1977

Carnap R.
Einführung in die Symbolische Logik mit besonderer Berüksichtigung ihrer Anwendungen
Springer, Wien - New York, 1973

Chellas B. F.
Modal Logic: an Introduction
Cambridge University Press, Cambridge, U. K. (1980)
Church A.
Introduction to Mathematical Logic,

Princeton, New Jersey, U.S.A., (1956) vol 1

D'Amelio C.
An Algebraic-logical Algorythm,
Metalogicon, **I**, 1, 3-17, (1988)

Grappone A. G.
Temporal, Modal, Psychoanalytical Functors in the Bivalent Logic,
Metalogicon, **I**, 2, 41-57, (1988)

Halmos P.
Lectures on Boolean Algebras
Springer-Verlag, New York Heidelberg Berlin (1974)

Lombardo Radice L.
Istituzioni di algebra astratta,
Feltrinelli, Milano, (1973[10])

Longo G.
Teoria dell'informazione
Boringhieri s.p.a., Torino, Italy, (1980)

Łukasiewicz J. and Tarski A.
Untersuchungen über den Aussagenkalkül,
Soc. Sci. Lett. Varsovie, **23**, (1930)

Malatesta M.
An Extension Of Gentzen's natural deduction
Metalogicon, **II**, 1, 1-32 (1989)

Matte Blanco I.
The Unconscious as Infinite Sets. An Essay in Bi-logic
G. Dukworth & Co. Ltd, London, UK, (1975)

Mendelson E.
Introduction to the Mathematical Logic,
D. Van Nostrand, Princeton, New Jersey, U.S.A., (1964)

Tarski A.
Der Wahrheitsbegriff in den formalisierten Sprachen,
Studia philos., **1**, 261-405, (1936)

Urquhart A.
The Undecidibility of Entailment and Relevant Implication,
JSL, **49**, 1059-1073, (1984)

Whithehead A. N. and Russell B.
Principia Mathematica,
Cambridge University Press, Cambridge, U.K., (1910^1, 1927^2)